Mathematical Tables *and other* Aids to Computation

A Quarterly Journal

Edited by

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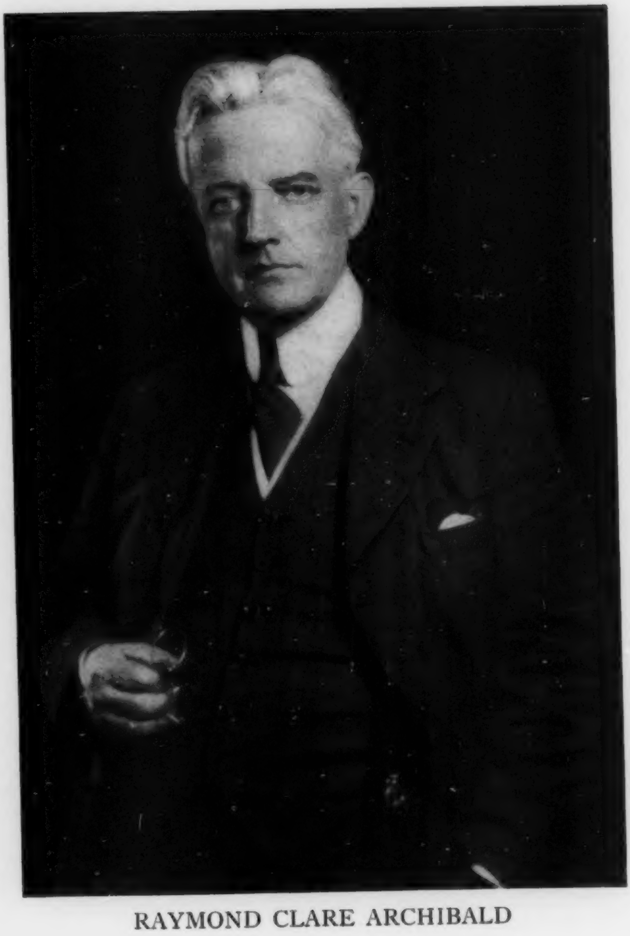

RAYMOND CLARE ARCHIBALD

Note of Appreciation

The conception of this quarterly journal, *Mathematical Tables and Other Aids to Computation* (*MTAC*) stemmed from the fertile mind of R. C. ARCHIBALD and its establishment was largely due to his initiative and perseverance. As its editor from its first issue in January, 1943, to the close of 1949, he has given freely of his time and effort. Its success in becoming a useful and often indispensable tool of reference for those concerned with computational problems is due largely to his unflagging interest and untiring efforts in an activity that claimed his well-nigh full-time devotion. The 28 issues published under his editorship will ever stand as a monument to his achievement.

On behalf of the National Research Council, I wish to record its thanks and appreciation to Dr. Archibald for the significant contribution he has made to a field that is of great interest throughout the Council.

As he lays down his editorial responsibilities, his many friends and colleagues extend to Dr. Archibald their warmest felicitations and, with his freedom from "copy deadlines," they wish for him opportunity to spend many years in the pursuit of his avocational interests.

R. C. GIBBS, Chairman
Division of Mathematical and
Physical Sciences
National Research Council

Raymond Clare Archibald

RAYMOND CLARE ARCHIBALD, Professor of Mathematics, retiring chairman of the Committee on Mathematical Tables and Other Aids to Computadion, 1939–49, founder and editor of *Mathematical Tables and Other Aids to Computation* of the National Research Council, 1943–49, is well known to the mathematicians of America. Born October 7, 1875 in South Branch, Stewiacke, Colchester County, Nova Scotia, he was a student in Sackville, New Brunswick, at Mt. Allison Academy (1885–89), University (1889–95, A.B. '94), and Ladies' College (1889–95, violin diplomas '94 and '95), where he developed marked ability in mathematics and the violin. He returned to the Ladies' College and from 1900–1907 was Professor of Violin, Harmony, History of Music, and Mathematics, while also serving as Librarian. Between 1895 and 1900 he studied at Harvard, where he received his bachelor's (1896) and master's (1897) degrees, at Berlin, and at the University of Strassburg where he was awarded his doctorate. After a year as Professor of Mathematics and chairman of the department at Acadia University, Professor Archibald came to Brown University in 1908, with which he has been associated ever since and where he is now Professor of Mathematics, Emeritus. He has been appointed delegate to many international congresses, mathematical and of wider scope; he has served as president of the Mathematical Association of America, of which he was a charter member, was twice vice-president of the American Association for the Advancement of Science (for Section A and for Section L respectively); was Librarian of the American

Mathematical Society 1921–41, member of the executive committee, Division of Mathematical and Physical Sciences of the National Research Council (1941–43) and member of many learned societies, here and abroad. His warm personal friends have included in particular SIMON NEWCOMB, the distinguished astronomer, and M. G. MITTAG-LEFFLER, of Sweden, one of the leading mathematicians of his day. He has been honored by institutions here and abroad of which may be mentioned Harvard University, Swiss Society of Naturalists, American Academy of Arts and Sciences, Bologna International Congress (1928); Academy of Work in Czecho-Slovakia; Academy of Sciences in Cluj, Rumania; University of Padua; Mathematical Association, England; Polish Mathematical Society. At Brown, in the face of great difficulties he has developed through years of unremitting care and intelligent effort, a Mathematical Library which has been described as "the most useful single source available in America." He has always been active in stimulating undergraduate interest in mathematics. He founded, with R. G. D. RICHARDSON and H. P. MANNING, in 1916 and maintained in unbroken continuity an undergraduate Mathematics Club at Brown, and organized and taught courses designed to broaden the appreciation and outlook of prospective teachers of mathematics. There is not room here to list in completeness his 265 numbered published writings, of which some headings cover many separate items. Readers of *MTAC* need not be reminded of all his contributions to this quarterly. No less intensive was his work for the *American Mathematical Monthly*, particularly from about 1918 to 1923, largely in connection with geometry, and the history of mathematics (his *Outline of the History of Mathematics* has come out in sixth edition, 1949). This brief sketch must close with merely a reference to some of the publications in which his work (other than books) has appeared. *Educational Times*, London; Royal Society of Canada, *Transactions; Notes and Queries*, London; *L'Intermédiare des Mathématiciens;* Edinburgh Mathematical Society, *Proceedings;* American Mathematical Society, *Bulletin; Mathematical Gazette; Science; American Mathematical Monthly;* National Academy of Sciences, *Memoires; Isis; Nature;* Congresso Internazionale dei Matematici, Bologna, *Atti*, 1928; *Encyclopaedia Britannica* (14th ed.); *Dictionary of American Biography; Scripta Mathematica;* American Academy of Arts and Sciences, *Proceedings; Osiris; Mathematics Teacher;* International Congress of Mathematicians, Oslo, *Proceedings*, 1937. He has also written several books, among them *Euclid's Book on Divisions of Figures*, published by the Cambridge University Press, 1915. Readers could consult any of twenty biographical sources for information concerning Professor Archibald, of which may be mentioned: *American Men of Science; Who's Who in America; Poggendorff* (v. 5 and v. 6); *Encyclopaedia Britannica* (14th ed.).

ALBERT A. BENNETT

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Checking by Differences—I

1. Introduction and Summary. When computing a table of numerical values of a mathematical function, an essential need is a check on the accuracy of the results. For this check to be fully satisfactory it must be independent, as nearly completely as possible, of the original calculations. This independence should apply to the method of computation used in the check and not only to the numerical details. Apart from this it is convenient to have a check that is as simple as possible to apply.

When the values computed form a systematic table for equally-spaced values of some associated variable—usually, but not necessarily, taken as argument—the best-known check is probably that provided by forming a table of differences. The accuracy of the results is then tested by an examination of the general run of the values of the differences of some high order, say 3rd, 5th or possibly 10th differences. It will be assumed in what follows that argument values are equidistant.

The check provided by this process of differencing is very easy to apply, and is almost always fully satisfactory in all the senses outlined above. The precise details of the process and its pitfalls do not, however, seem to have been set out fully in print. It is the purpose of this paper to consider some aspects of the process, and to discuss possible methods of detecting and finding several types of error.

In **2**, the normal difference table for equal argument-intervals is discussed. The cases of a polynomial and of a general function are both considered.

In **3**, the effect of an isolated error is exhibited, and in **4**, methods for distinguishing true errors or blunders from inevitable rounding-off errors are considered.

It is proposed to examine in a later paper some cases where there are blunders due to causes other than mistakes in function values, or where there are coupled or systematic blunders, or where the resulting effects are overlapping for other reasons. In particular, ways of distinguishing mistakes made during the formation of differences are not considered in the present paper.

2. The Normal Difference Table. 2.1. When a table of exact values of a polynomial of degree n, for equidistant values of the argument, is differenced, the values of the n-th differences are all equal, and values of higher differences are all zero. This is too well known for a numerical illustration to be needed.

If, however, the values of the polynomial tabulated are rounded off to a fixed number of decimals, the n-th differences are no longer constant, but periodic, with period depending on the degree of the polynomial and on the number of figures dropped. Higher differences also form cycles of the same period. This may be illustrated by means of the quadratic function $10x(x - 1)$. If this is tabulated to the nearest integer for interval .1 in x, the second differences run through the ten values 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, this cycle being repeated indefinitely. It may be noted that the average value is .2, in agreement with the true second difference for a table of exact values. Likewise, the third differences give a cycle 0, 0, $+1$, -1, 0, 0, 0, 0, $+1$, -1, with average zero.

2.2. Consider now the more usual case of a function that cannot be tabulated exactly. Table I gives 5-decimal values of $\log_{10} N$, for $N = 10(1)30$, with all differences to order 10. In this illustration the interval of the argument has been chosen to be small enough for checking by differences to be feasible. If too large an interval is chosen, the differences may diverge as order increases.

It will be noted that:

(i) For low orders of differences, regularity is apparent. The magnitude diminishes as order increases. This holds up to about δ^4 in the present table.

(ii) For high orders of differences, irregularity appears. This is due to the inevitable rounding-off errors, and shows first in an irregular sign-pattern. Later, as order increases, the differences increase in magnitude but more or less irregularly.

(iii) For a sequence of differences of a particular (high) order, the larger values tend to occur in groups, with signs strictly alternating and with values falling away on either side from a central maximum value, or pair of such values.

TABLE I

N	Log N		δ^2		δ^4		δ^6		δ^8		δ^{10}
10	1.00000										
		+4139									
11	1.04139		−360								
		+3779		+57							
12	1.07918		−303		−11						
		+3476		+46		−1					
13	1.11394		−257		−12		+ 9				
		+3219		+34		+8		−20			
14	1.14613		−223		− 4		−11		+37		
		+2996		+30		−3		+17		−61	
15	1.17609		−193		− 7		+ 6		−24		+ 90
		+2803		+23		+3		− 7		+29	
16	1.20412		−170		− 4		− 1		+ 5		− 28
		+2633		+19		+2		− 2		+ 1	
17	1.23045		−151		− 2		− 3		+ 6		− 10
		+2482		+17		−1		+ 4		− 9	
18	1.25527		−134		− 3		+ 1		− 3		+ 8
		+2348		+14		0		+ 1		− 1	
19	1.27875		−120		− 3		+ 2		− 4		+ 4
		+2228		+11		+2		− 3		+ 3	
20	1.30103		−109		− 1		− 1		− 1		+ 17
		+2119		+10		+1		− 4		+20	
21	1.32222		− 99		0		− 5		+19		− 75
		+2020		+10		−4		+15		−55	
22	1.34242		− 89		− 4		+10		−36		+132
		+1931		+ 6		+6		−21		+77	
23	1.36173		− 83		+ 2		−11		+41		−153
		+1848		+ 8		−5		+20		−76	
24	1.38021		− 75		− 3		+ 9		−35		+133
		+1773		+ 5		+4		−15		+57	
25	1.39794		− 70		+ 1		− 6		+22		− 83
		+1703		+ 6		−2		+ 7		−26	
26	1.41497		− 64		− 1		+ 1		− 4		
		+1639		+ 5		−1		+ 3			
27	1.43136		− 59		− 2		+ 4				
		+1580		+ 3		+3					
28	1.44716		− 56		+ 1						
		+1524		+ 4							
29	1.46240		− 52								
		+1472									
30	1.47712										

TABLE II

TABLE II

N	E		δ^2		δ^4		δ^6		δ^8		δ^{10}
10	0.00										
		+.27									
11	+ .27		−.41								
		−.14		+ .77							
12	+ .12		+.35		−1.86						
		+.21		−1.10		+4.56					
13	+ .34		−.74		+2.69		−10.15				
		−.53		+1.60		−5.59		+20.42			
14	− .20		+.85		−2.90		+10.27		−37.17		
		+.32		−1.30		+4.68		−16.75		+61.07	
15	+0.13		−.45		+1.78		− 6.48		+23.90		− 90.03
		−.13		+ .47		−1.80		+ 7.16		−28.96	
16	− .00		+.02		− .03		+ .68		− 5.06		+ 27.98
		−.11		+ .44		−1.13		+ 2.10		− .98	
17	− .11		+.46		−1.16		+ 2.78		− 6.03		+ 9.99
		+.36		− .71		+1.65		− 3.93		+ 9.01	
18	+ .25		−.25		+ .49		− 1.16		+ 2.98		− 8.00
		+.11		− .22		+ .49		− .96		+ 1.01	
19	+ .36		−.47		+ .98		− 2.11		+ 3.99		− 4.00
		−.36		+ .76		−1.62		+ 3.03		− 3.00	
20	−0.00		+.29		− .64		+ .92		+ .99		− 17.00
		−.07		+ .12		− .71		+ 4.02		−20.00	
21	− .07		+.41		−1.35		+ 4.94		−19.01		+ 75.00
		+.34		−1.23		+4.23		−14.98		+55.00	
22	+ .27		−.82		+2.88		−10.05		+36.00		−132.00
		−.48		+1.65		−5.82		+21.01		−77.00	
23	− .22		+.83		−2.94		+10.96		−41.00		+153.00
		+.34		−1.29		+5.15		−19.99		+76.00	
24	+ .12		−.46		+2.21		− 9.03		+35.00		−133.00
		−.12		+ .92		−3.88		+15.01		−57.00	
25	+0.00		+.46		−1.67		+ 5.98		−22.00		+ 83.00
		+.33		− .75		+2.10		− 7.00		+26.00	
26	+ .33		−.29		+ .43		− 1.02		+ 4.00		
		+.04		− .32		+1.08		− 3.00			
27	+ .38		−.61		+1.51		− 4.01				
		−.57		+1.18		−2.93					
28	− .20		+.57		−1.43						
		−.00		− .24							
29	− .20		+.33								
		+.33									
30	+0.13										

The properties (ii) and (iii) are emphasized if a true error, or "blunder,"[1] should occur, and form the basis of the method for the detection and location of such blunders by differencing.

These observations are readily explained by noting that each value of the function is the sum of a tabular entry and a rounding-off error. This rounding-off error forces the value tabulated to be a multiple of the unit of the final decimal, and is at most half of this unit in magnitude.

Table II shows the rounding-off errors E of Table I, with differences to the 10th. These differences were all obtained with three further figures and rounded off individually, and are within half a unit of the 7th decimal from the true value.

It will be noted that as the order of the differences increases beyond the 4th—that is, from the point in Table I where the differences begin to show obvious irregularity—the differences in Table II tend to be more and more nearly integral multiples of the 5th decimal unit, and to give values approaching zero more and more closely when added to the corresponding values in Table I. This confirms the expectation that the differences of the true values of the function tabulated continue to decrease, with increasing order of differences, beyond the order to which this decrease remains apparent in a 5-decimal table.

TABLE III

f		δ^2		δ^4	
0		0		0	
	0		0		0
0		0		0	
	0		0		$+\epsilon$
0		0		$+\epsilon$	
	0		$+\epsilon$		-5ϵ
0		$+\epsilon$		-4ϵ	
	$+\epsilon$		-3ϵ		$+10\epsilon$
ϵ		-2ϵ		$+6\epsilon$	
	$-\epsilon$		$+3\epsilon$		-10ϵ
0		$+\epsilon$		-4ϵ	
	0		$-\epsilon$		$+5\epsilon$
0		0		$+\epsilon$	
	0		0		$-\epsilon$
0		0		0	
	0		0		0
0		0		0	

3. An Isolated Error. 3.1. Consider next the effect of an isolated error or blunder ϵ. This is exhibited in Table III. This extends only to 5th differences, as it is sufficiently obvious that the coefficients are binomial coefficients with alternating signs.

3.2. The method for detecting, locating and evaluating a blunder in a table of exact values of a polynomial is now clear. A difference table is formed and the differences of some order p greater than n, the degree of the polynomial, are examined. These differences should all be zero; if, however, it is observed that some are not zero but alternate in sign and have magnitudes proportional to the binomial coefficients

$$\binom{p}{r} = \frac{p!}{r!(p-r)!}$$

then an isolated error is indicated.

If $p = 2k$ is even, there will be a pth difference that is numerically greater than the others; the blunder should be found[2] in the function value on a level with this. Denote this largest difference, with its sign attached, by D_{2k}. The value of the amount to be added to the function value in question, in order to correct it, is one of the values

$$+D_2/2,\ -D_4/6,\ +D_6/20,\ -D_8/70,\ +D_{10}/252,\ \cdots,\ (-1)^{k-1}D_{2k}\Big/\binom{2k}{k}$$

It should be noted that these apply only when the corresponding difference D_{2k} should be zero, i.e., when $2k > n$, the degree of the polynomial concerned.

If $p = 2k + 1$ is odd, there will be two successive pth differences of equal magnitude, larger than the rest. The blunder should be found in the function value at the level half-way between these. If the upper of these differences is D_{2k+1} and the lower $-D_{2k+1}$, the correction to be added to the function value is one of the values (with $2k + 1 > n$)

$$-D_1,\ +D_3/3,\ -D_5/10,\ +D_7/35,\ -D_9/126,\ \cdots,\ (-1)^k D_{2k-1}\Big/\binom{2k-1}{k}$$

3.3. Consider next a difference table involving a function that is not a polynomial. In this case the effect due to a true error or blunder, exceeding half a final unit, is mixed up with the effects of the rounding-off errors. Detection and location of the error involve the disentanglement of these effects.

For large blunders the method of detection is, with small modifications, the same as for a polynomial. A difference table is first formed to an order p of differences such that the *normal* vertical sequence of signs (i.e., the sequence of signs in a region free from the effects of blunders) has ceased to be regular. This means that the pth differences of the true function—which are regular—are swamped by the irregularities due to the rounding-off errors, that is, that the function differences are effectively zero.[3] Any sequence of differences having the numerically greatest difference substantially larger than the normal pth differences due to rounding-off will at once stand out, and the error indicated can be located and estimated in the same manner as in **3.2**, except that:

(i) The successive differences will be only approximately proportional to the binomial coefficients of order p.

(ii) Instead of estimating the correction from a single value D_p of the pth difference, it is better to add the numerical values of a sequence of differences, centered about the largest, and to divide by the sum of the corresponding binomial coefficients. It is also sometimes useful to repeat and verify the estimate with differences of a higher order, where true function differences will usually be smaller.

Table IV illustrates these various points.

An error is apparent in δ^2, but, away from the neighborhood of the error, the signs are regular in δ^3 and even in δ^4, as may be seen in Table I. In δ^5 the large differences are in the ratios -10, $+10$, -5. Thus, approximately, the correction C is given by $(10 + 10 + 5)C = 180 + 182 + 89 = 451$. Hence $25C = 451$ and $C = 18$. Hence, log 19 should read 1.27875, agreeing with Table I.

Use of a single value of δ^7 gives

$$(10 + 2\cdot 10 + 5)C = 180 + 2\cdot 182 + 89 = 633$$

giving, again, $C = 18$.

Choice of suitable difference to which to apply the process is determined by the equality of results from differences of two successive orders; this equality is taken to indicate that the variation of the result with increasing order of difference has ceased.

A process for filling in the gap that appears, at first sight, more satisfactory is to use LAGRANGE's interpolation formula based on tabular values but omitting, of course, the value needing correction. The gap should be as near the middle of the run of values as possible. If, however, p points are used, Lagrange's formula assumes that the p-th difference is zero; that is, the result will be precisely that obtained by equating to zero the appropriate p-th difference in order to determine the error.

TABLE IV

N	$\log N$		δ^2		δ^4	
16	1.20412					
		+2633				
17	1.23045		−151			
		+2482		− 1		
18	1.25527		−152		+ 69	
		+2330		+68		−180
19	1.27857		− 84		−111	
		+2246		−43		+182
20	1.30103		−127		+ 71	
		+2119		+28		− 89
21	1.32222		− 99		− 18	
		+2020		+10		+ 14
22	1.34242		− 89		− 4	
		+1931		+ 6		
23	1.36173		− 83			
		+1848				
24	1.38021					

4. Disentanglement of Blunders from Rounding-off Errors. 4.1. It is desirable to know a lower limit to the size of blunder that can be detected with certainty, and an upper limit to the size of those blunders for which it is almost useless to attempt detection by differencing. Blunders intermediate in size may or may not be detected, depending on the run of neighboring rounding-off errors; it is useful to have an estimate of the probability of detecting such an intermediate blunder according to its size. A more immediate problem, however, is the determination of the probability that a difference of given order and given size is due solely to the effect of rounding-off errors. The complementary probability gives the likelihood of a blunder. These limits and probabilities depend, of course, on the order of the differences examined.

In practice the procedure is as follows: All the differences of a particular, sufficiently high, order are examined. Those numerically exceeding the limit L that indicates a blunder with certainty (there may be several such large differences in succession, of alternating signs, due to a single blunder) are noted and examined carefully in order to locate the blunder, which must then be removed. When all such blunders have been eliminated there re-

mains a run of differences, none numerically larger than L, but which may have some entries, larger than the majority, that could arise from an unlikely combination of rounding-off errors, but have a good chance of being due to a small end-figure blunder. The nearer such a difference is to L in magnitude, the more nearly does the probability that it is due to a blunder approach unity. The problem, then, is to choose K such that all differences numerically greater than K should be examined, while all those numerically not greater than K may be accepted as satisfactory, being almost certainly due to rounding-off errors.

4.2. It is easy to determine the limit L above which a blunder is certainly indicated. The sequence of rounding-off errors giving rise to the greatest possible effect in the differences is $\cdots + \frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, \cdots$ extending indefinitely in both directions. This leads to the sequence $\cdots, +2^{p-1}, -2^{p-1}, +2^{p-1}, -2^{p-1}, \cdots$ in the pth differences. The maximum rounding-off effect L in the pth difference is thus 2^{p-1} in magnitude.

On the other hand, if a blunder ϵ (assumed positive and not too small) is made in a tabular value, the case most unfavorable for detection comes from the sequence

$$(S) \qquad \cdots+\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2}, \epsilon, +\tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2}, \cdots$$

The corresponding differences on a level with the error ϵ are given in the second line of Table V.

TABLE V

Difference	2nd	4th	6th	8th	10th	12th
From sequence (S)	$1-2\epsilon$	$6\epsilon-5$	$22-20\epsilon$	$70\epsilon-93$	$386-252\epsilon$	$924\epsilon-1586$
Numerically largest legitimate errors $\pm L$	-2	$+8$	-32	$+128$	-512	$+2048$
Maximum error ϵ_{max} that can escape detection	3/2	13/6	27/10	221/70	449/126	1817/462
	1.50	2.17	2.70	3.16	3.56	3.93

If these are just equal, numerically, to the maximum legitimate values due to rounding-off (given in the third line of Table V, with the appropriate sign to give maximum ϵ), then the corresponding maximum errors that might just escape detection result. These are given in the fourth and fifth lines of the table. In fact, for order $2k$,

$$\epsilon_{max} = -\tfrac{1}{2} + 4^k \Big/ \binom{2k}{k}.$$

Larger blunders cannot escape detection.

4.3. The limits L and the blunders ϵ_{max} that may just escape detection are, however, sometimes too great to be of practical use. Differences, due entirely to rounding-off, with magnitude approaching L, are so rarely met with that the occurrence of such a difference is a strong reason for suspecting a blunder. It is necessary, then, to choose a different limit K, as indicated in 4.1.

Satisfactory practical limits K have been obtained, from experience in the examination of many tables, by Dr. L. J. Comrie. These limits are very roughly such that about 1 difference in 100 exceeds K numerically and requires more careful examination, and are as follows:

Difference	3rd	4th	5th	6th	8th	10th	12th	15th
Practical limit K	3	6	12	22	80	300	1100	8000

The determination of exact theoretical probabilities for differences of various sizes is a matter of some difficulty. A theory and technique have been devised, but results are not yet complete.[4] If we wish to choose the limit K so that the chance of a difference arising from rounding-off errors that exceeds K is less than .01, while the chance of an error exceeding $K - 1$ is greater than .01, the following results are relevant.

Order of Difference	Num. Value of Difference	Chance of Occurrence	Order of Difference	Num. Value of Difference	Chance of Occurrence
2	$\geqslant 1$	0.5	6	$\geqslant 21$	0.0128
	$\geqslant 2$	0.0		$\geqslant 22$	0.0079
3	$\geqslant 3$	0.04	7	$\geqslant 41$	0.0108
	$\geqslant 4$	0.00		$\geqslant 42$	0.0084
4	$\geqslant 6$	0.0130	8	$\geqslant 79$	0.0111
	$\geqslant 7$	0.0009		$\geqslant 80$	0.0099
5	$\geqslant 11$	0.0140	9	$\geqslant 155$	0.0103
	$\geqslant 12$	0.0052		$\geqslant 156$	0.0097

The probabilities serve to show the consistency of the practical limits given above, and to provide additional limits of 42 for the 7th difference and 156 for the 9th difference.

4.4. It is not to be supposed that the limits K of the last section must be adhered to rigidly. The major field for use of these precise limits is for differencing tables with a final figure that should be correct within half a unit. In this case, the original calculations will contain one or more extra figures, and these extra figures should be used in the examination of the one doubtful case in 100 previously mentioned, in order to verify that the actual rounding-off errors that occur do give rise to a difference of the right sign and about the right size.

If a printed table is differenced, the extra figures may not be available, while if the function is one difficult to compute, and if the table is a long one, the work of recomputing values to test one difference in 100 may be prohibitive. In such cases it may be necessary to adopt a higher limit than K, possibly even L may have to be used, in which case one would state, for example, that an examination of the 8th differences showed that no isolated end-figure error of 3.2 units or more could occur in the table. The possibility of systematic or coupled blunders remains.

An alternative plan is to difference the function values as computed, retaining all figures computed, including one or more guard figures. It is then unnecessary to examine marginal blunders or errors too closely, and the limit L, or even higher limits such as $2L$ or $3L$, might be adopted. This procedure has the advantage that blunders large enough to need correction will stand out prominently.

5. Part I of this paper has been concerned with the location and detection of isolated blunders. There remain several possibilities to be discussed in Part II. These include:

(i) The recognition of blunders made during the differencing.

(ii) The detection and location of coupled or multiple blunders, such as

(a) two equal blunders in successive values or (b) a systematic succession of erroneous values in a table.

It is also proposed to give error patterns, such as that in Table III, for tables of *divided* differences, for use with tables having certain common arrangements of arguments at unequal intervals, for example, with a table having arguments

$$0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, 1, 1\tfrac{1}{4}, 1\tfrac{1}{3}, \cdots$$

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[1] The introduction of this useful distinction in name between rounding-off and true errors is due to C. R. G. Cosens.

[2] It must be remarked that the sequences of errors discussed here can arise from a cause other than the one indicated, though such causes are comparatively less common. For instance, if differencing is done on a calculating machine, a function value may be correctly recorded, but wrongly set on the machine. Likewise, a different sequence of differences indicates blunders of a different type. It is hoped to discuss some of these in Part II of the paper.

[3] In practice, a large blunder shows up well enough for location in earlier orders of differences, in fact, as soon as the largest of the differences due to the blunder sufficiently exceeds the true differences in magnitude, say in the ratio 5 to 1 or 10 to 1. Detection is possible in still earlier differences.

[4] A. van Wijngaarden & W. L. Scheen of the Mathematisch Centrum of Amsterdam, Holland, have developed the theory independently and have obtained an asymptotic expansion. The result given for 9-th differences in our table was obtained by them and communicated to us for inclusion in this paper. Their 1 percent limit for 10-th differences is 303.

An ENIAC Determination of π and e to more than 2000 Decimal Places

Early in June, 1949, Professor John von Neumann expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits, suggesting the employment of one of the formulas:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$
$$\pi/4 = 8 \arctan 1/10 - 4 \arctan 1/515 - \arctan 1/239$$
$$\pi/4 = 3 \arctan 1/4 + \arctan 1/20 + \arctan 1/1985$$

in conjunction with the Gregory series

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} x^{2n+1}.$$

Further interest in the project on π was expressed in July by Dr. Nicholas Metropolis who offered suggestions about programming the calculation.

Since the possibility of official time was too remote for consideration, permission was obtained to execute these projects during two summer holiday week ends when the ENIAC would otherwise stand idle, and the planning and programming of the projects was undertaken on an extra-curricular basis by the author.

The computation of e was completed over the July 4th week end as a

practice job to gain experience and technique for the more difficult and longer project on π. The reciprocal factorial series was employed:

$$e = \sum_{n=0}^{\infty} (n!)^{-1}.$$

The first of the above-mentioned formulas was employed for the computation of π; its advantage over the others will be explained later. The computation of π was completed over the Labor-Day week end through the combined efforts of four members of the ENIAC staff: CLYDE V. HAUFF (who checked the programming for π), Miss HOMÉ S. MCALLISTER (who checked the programming for e), W. BARKLEY FRITZ and the author, taking turns on eight-hour shifts to keep the ENIAC operating continuously throughout the week end.

While the programming for e is valid for a little over 2500 decimal places and, with minor alterations, can be extended to much greater range, and while the programming for π is valid for around 7000 decimal places, the arbitrarily selected limit of 2000+ was a convenient stopping point for e and about all that could be anticipated for a week end's operation for π.

While the details of the programming for each project were completely different, the general pattern of procedure was roughly the same, and both projects will be discussed together. In both projects the ENIAC'S divider was employed to determine a chosen number i of digits of each successive term of the series being computed, the remainder after each division being stored in the ENIAC'S memory and the digits of each term being added to (or subtracted from) the cumulative total. After performing this operation for as many successive terms as practicable, the remainders for these terms were printed on an I.B.M. card (the standard input-output vehicle for the ENIAC), and the process was repeated, continuing through some term beyond which the digits of and remainders for all further terms would be zeros. At this point was printed the cumulative total of the digits of the individual terms, which yielded (after adjustment for carry-over) the actual digits of the series being determined.

The cards bearing the remainders then were fed into the ENIAC reader, and the entire process was repeated for the next i digits, the ENIAC reading each remainder in turn and placing it before the digits of the appropriate term. Each deck of cards bearing remainders was then employed to determine the "next" i digits and the "next" deck of "remainder" cards continuing through the first stopping point beyond the 2000th decimal place. The cards bearing the cumulative totals of sets of i digits of the terms were then adjusted for carry-over into each preceding set of i digits. In the case of e this yielded the final result; in the case of π all the above described operations were performed once for each inverse tangent series, so that each set of "cumulative total" cards, adjusted for carry-over, yielded the digits of one of the series, the final result being determined by the combination of these series in appropriate manner.

The number of places i chosen for each interval of computation, the maximum magnitude of each remainder, the amount of memory space available, and the details of divider operation (the number of places to which division can be performed to yield a positive remainder, and the necessary conditions of relative and absolute positioning of numerator and

denominator) all were interrelated, and where opportunity for selection existed, that selection was made which provided maximum efficiency of computation. In the case of π there was imposed the additional requirement that identical programming apply for all series employed, and for this reason the formula:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$

was superior to the other two.

In order to insure absolute digital accuracy, the programming was arranged so that one half applied to computation and the other half to checking. Before any deck of "remainder" cards was employed to determine the next i digits, the cards were reversed and employed in the checking sequence to confirm each division by a multiplication and each addition by a subtraction and vice versa, reproducing the previous deck of "remainder" cards and insuring that the cumulative total reduced to zero. (In the case of e this was a simple inversion of the computation; in the case of π the factor $(2n + 1)^{-1}$ in each term made it a more complicated affair.) After the correctness of each deck was established through this checking, the "remainder" cards were rereversed, and the computation proceeded for the next i digits.

Since the determination of each i digits was not begun until the determination of the previous i digits had been confirmed by checking, the ENIAC stood idle during the reversals and rereversals and comparisons of the decks in the computation of e; in the case of π, however, the ENIAC was never idle, for operation on each series was alternated with operation on the other, card-handling on either being accomplished while the other was being operated upon by the ENIAC. In the case of e, insurance against any undiscovered accidental misalignment of cards was provided by rerunning the entire computation without checking, i.e., without card reversals, confirming the original results; in the case of π, the same assurance was provided by a programmed check upon the identification numbers of each successive card in both computation and checking.

In the case of e, there was printed (in addition to each "remainder" card) a card containing the current i digits of $(n!)^{-1}$ for $n = 20K$; $K = 1, 2, 3 \cdots$; in the case of π only remainder and final total cards were printed.

The ENIAC determinations of both π and e confirm the 808—place determination of e published in *MTAC*, v. 2, 1946, p. 69, and the 808—place determination of π published in *MTAC*, v. 2, 1947, p. 245, as corrected in *MTAC*, v. 3, 1948, p. 18–19. 7

Only the following minor observation is offered at this time concerning the randomness of the distribution of the digits. Publication on this subject will, however, be forthcoming soon. A preliminary investigation has indicated that the digits of e deviate significantly from randomness (in the sense of staying closer to their expectation values than a random sequence of this length normally would) while for π no significant deviations have so far been detected.

The programming was checked and the first few hundred decimal places of each constant were determined on a Sunday before each holiday week end mentioned above, the principal effort being made on the longer week end. The actual required machine running time for both computation and checking in the case of e was around 11 hours, though card-handling time approxi-

mately doubled this, and the recomputation without checking added about 6 hours more; actual required machine running time (including card-handling time) for π was around 70 hours.

The following values of π and e have been rounded off to 2035D and 2010D respectively.

π =	3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
	58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
	82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
	48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
	44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
	45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
	72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
	78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
	33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
	07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
	98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
	60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
	00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
	14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
	42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
	51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
	50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
	71010	00313	78387	52886	58753	32083	81420	61717	76691	47303
	59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
	18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
	38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
	03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
	55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
	81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
	85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
	94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
	93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
	25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
	67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
	55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
	32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
	63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
	81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
	16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
	45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
	56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
	82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
	73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
	06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
	46776	46575	73962	41389	08658	32645	99581	33904	78027	59009
	94657	64078	95126	94683	98352	59570	98258			

e =	2.71828	18284	59045	23536	02874	71352	66249	77572	47093	69995
	95749	66967	62772	40766	30353	54759	45713	82178	52516	64274
	27466	39193	20030	59921	81741	35966	29043	57290	03342	95260
	59563	07381	32328	62794	34907	63233	82988	07531	95251	01901
	15738	34187	93070	21540	89149	93488	41675	09244	76146	06680
	82264	80016	84774	11853	74234	54424	37107	53907	77449	92069
	55170	27618	38606	26133	13845	83000	75204	49338	26560	29760

67371	13200	70932	87091	27443	74704	72306	96977	20931	01416
92836	81902	55151	08657	46377	21112	52389	78442	50569	53696
77078	54499	69967	94686	44549	05987	93163	68892	30098	79312
77361	78215	42499	92295	76351	48220	82698	95193	66803	31825
28869	39849	64651	05820	93923	98294	88793	32036	25094	43117
30123	81970	68416	14039	70198	37679	32068	32823	76464	80429
53118	02328	78250	98194	55815	30175	67173	61332	06981	12509
96181	88159	30416	90351	59888	85193	45807	27386	67385	89422
87922	84998	92086	80582	57492	79610	48419	84443	63463	24496
84875	60233	62482	70419	78623	20900	21609	90235	30436	99418
49146	31409	34317	38143	64054	62531	52096	18369	08887	07016
76839	64243	78140	59271	45635	49061	30310	72085	10383	75051
01157	47704	17189	86106	87396	96552	12671	54688	95703	50354
02123	40784	98193	34321	06817	01210	05627	88023	51930	33224
74501	58539	04730	41995	77770	93503	66041	69973	29725	08868
76966	40355	57071	62268	44716	25607	98826	51787	13419	51246
65201	03059	21236	67719	43252	78675	39855	89448	96970	96409
75459	18569	56380	23637	01621	12047	74272	28364	89613	42251
64450	78182	44235	29486	36372	14174	02388	93441	24796	35743
70263	75529	44483	37998	01612	54922	78509	25778	25620	92622
64832	62779	33386	56648	16277	25164	01910	59004	91644	99828
93150	56604	72580	27786	31864	15519	56532	44258	69829	46959
30801	91529	87211	72556	34754	63964	47910	14590	40905	86298
49679	12874	06870	50489	58586	71747	98546	67757	57320	56812
88459	20541	33405	39220	00113	78630	09455	60688	16674	00169
84205	58040	33637	95376	45203	04024	32256	61352	78369	51177
88386	38744	39662	53224	98506	54995	88623	42818	99707	73327
61717	83928	03494	65014	34558	89707	19425	86398	77275	47109
62953	74152	11151	36835	06275	26023	26484	72870	39207	64310
05958	41166	12054	52970	30236	47254	92966	69381	15137	32275
36450	98889	03136	02057	24817	65851	18063	03644	28123	14965
50704	75102	54465	01172	72115	55194	86685	08003	68532	28183
15219	60037	35625	27944	95158	28418	82947	87610	85263	98139
55990	06738								

Values of the auxiliary numbers arccot 5 and arccot 239 to 2035D are in the possession of the author and also have been deposited in the library of Brown University and the UMT FILE [1] of *MTAC.*

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[1] See *MTAC,* v. 4, p. 29.

RECENT MATHEMATICAL TABLES

691[A].—M. LOTKIN, "Table of the first 200 factorials to 20 places," Ballistic Research Laboratories, Aberdeen Proving Ground, *Technical Note* no. 106, 1949, 11 p. mimeograph, 21.7 × 27.8 cm.

The table gives the first 20 significant figures of $n!$ for $n = 1(1)\ 200$ together with the exponent of the power of 10 by which the figure should be multiplied to give the approximate value of $n!$ The author was unaware of a previous table by UHLER[1] giving the exact values of these factorials.

Professor Uhler reports that a comparison shows that the present tables are quite without error. For references to other tables of large factorials see *MTAC* v. 1, p. 125, 163, 312, 452, v. 3, p. 205, 340, 355.

[1] H. S. UHLER, *Exact values of the first 200 factorials*, New Haven, 1944. [*MTAC*, v. 1, p. 312].

692[A].—H. S. UHLER, "The Arabian Nights' factorial and the weighted mean factorial," *Scripta Math.*, v. 15, 1949, p. 94–96.

This note gives the values of $450!\cdot 10^{-111}$ and $448!\cdot 10^{-109}$. The number 450! has 1001 digits, hence the title. The author gives the frequency of each digit 0–9 in the 890 digits of $450!\cdot 10^{-111}$ from which we deduce that the probability of obtaining such a distribution from a wholly random sequence of digits is a little less than 1/5. For other large factorials computed by the author and others see the preceding review.

693[C, E, K].—K. M. MATHER, "The analysis of extinction time data in bioassay," *Biometrics*, v. 5, 1949, p. 127–143.

This paper contains two tables. Table I (p. 136–137) gives $\ln \ln (1/x)$ for $x = 0(.001).999$ to 3D. Table II (p. 138–139) gives 4D values of $x + e^{-x}$, $-e^{-x} \exp (e^x)$, $e^{2x}/[\exp (e^x) - 1]$ for $x = -5(.1)-2(.05)+1(.1)1.9$. The values of $-e^{-x} \exp (e^x)$ are nearly all incorrect and appear to have been computed in a very casual manner. For example, $x = 0$, the value $-e$ is given as -2.7181. Other errors are by no means confined to the last decimal. For example for $x = 1.9$, the author has -114.9425 instead of -119.8085 and for $x = -4.6$ the author has -100.0000 in lieu of -99.4843. This table should not be trusted beyond 3 significant figures.

D. H. L.

694[C].—C. S. SMITH, "The intercommunication of atomic and weight percentages," p. 196–199 of *Metals Handbook, 1948 edition*, Cleveland, Ohio, 20.7 × 27.8 cm.

On p. 198 are two 4D tables of $10 + \log [x/(100 - x)]$ covering the range $x = 0(.01)5(.1)94.9$.

R. C. A.

695[F].—F. V. ATKINSON & LORD CHERWELL, "The mean-value of arithmetic functions," *Quart. Jn. Math.*, v. 20, 1949, p. 65–79.

On p. 76 there is a table of the number of k-th power free numbers $<250\,000$ of the form $n^k + h$ for $k = 3, 4, 5$ and $h = 1, 2, 3$ together with the corresponding values obtained from an approximate formula.

696[F].—J. LEHNER, "Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$," *Amer. Jn. Math.*, v. 71, 1949, p. 373–386.

The function j may be defined by

$$\begin{aligned} j &= x^{-1}\{1 + 240 \sum_{m=1}^{\infty} m^3x^m(1 - x^m)^{-1}\}^3 \prod_{n=1}^{\infty} (1 - x^n)^{-24} \\ &= x^{-1} + 744 + 196884x + 21493760x^2 + \cdots = \sum_{n=-1}^{\infty} c(n)x^n. \end{aligned}$$

Although this fundamental function was first investigated by FELIX KLEIN half a century ago, it is only in recent years that some attention has been paid to the properties of the coefficients $c(n)$. A small table of $c(n)$ for $n = 0(1)24$ has been given by ZUCKERMAN.[1] From this table the author has derived a table (p. 384) showing the highest power of p dividing $c(n)$ for $p = 2, 3, 5, 7, 11$ and $n = 1(1)24$. The additional value

$$c(25) = 12\ 18832\ 84330\ 42251\ 04333\ 51500,$$

given by LEHMER,[2] produces

25 | 2 ,3 ,3 ,0 ,0

as the 25th line of the table, as noted by the author (p. 386).

D. H. L.

[1] H. S. ZUCKERMAN, "The computation of the smaller coefficients of $J(\tau)$," Amer. Math. Soc., *Bulletin*, v. 45, 1939, p. 917–919.
[2] D. H. LEHMER, "Properties of the coefficients of the modular invariant $J(\tau)$," *Amer. Jn. Math.*, v. 64, 1942, p. 488–502, (p. 491).

697[F].—A. V. PRASAD, "A non-homogeneous inequality for integers in a special cubic field, I." K. Acad. van Wetenschappen, Amsterdam, *Proc.*, v. 52, 1949, p. 240–250. *Indagationes Math.*, v. 11, 1949, p. 55–65.

On p. 247(62) there is a 6D table of n-th powers of $\theta = 1.32471795$, where θ is the real root of $\theta^3 - \theta - 1 = 0$ for $n = -7(1) - 2(\frac{1}{2})4, 5$.

698[F].—H. C. ROBERT, "Pythagorean triangles and their inscribed circles," *Duodecimal Bulletin*, v. 5, 1949, p. 41–46. 13.8 × 21.5 cm.

A table (p. 44–46) is given of right triangles with integral sides arranged according to the radius R of the inscribed circle for $R = 1(1)17$. In addition to the sides of the triangle, the perimeter and the pythagorean generators are given. Of the 74 triangles listed, 31 are primitive. The table is given in duodecimal notation.

699[F].—P. VARNAVIDES, "On the quadratic form $x^2 - 7y^2$," R. Soc. London, *Proc.*, v. 197A, 1949, p. 256–268.

On p. 259 there is a table of 10 integers in the field $k(\sqrt{7})$ which are particularly small in absolute value.

700[F].—PIET WIJDENES, *Beginselen van de Getallenleer*. Second ed., Groningen Noordhoff N.V., 1949, 260 p. 15.5 × 24.2 cm. Paper cover 8.25 florins; bound 10.50. The first edition, 236 p. appeared in 1937.

On p. 218–224 is a factor table of numbers less than 20000 not divisible by 2, 3, 5, 7, 11.

701[F].—D. YARDEN, "Table of the distribution of zeros in the period mod p of a recurring sequence of order 3," (Hebrew) *Riveon Lemat.*, v. 2, 1948, p. 65–66.

Four recurring series are involved in this note [*MTAC*, v. 3, p. 519]

$$U_n = U_{n-2} + U_{n-3} \qquad V_n = V_{n-2} + V_{n-3}$$
$$\bar{U}_n = -\bar{U}_{n-2} + \bar{U}_{n-3} \qquad \bar{V}_n = -\bar{V}_{n-2} + \bar{V}_{n-3}$$

with initial conditions

$$(U_0, U_1, U_2) = (\bar{U}_0, \bar{U}_1, \bar{U}_2) = (0, 0, 1)$$
$$(V_0, V_1, V_2) = (\bar{V}_0, \bar{V}_1, -\bar{V}_2) = (3, 0, 2).$$

The tables give for each series the values of $n(\mathrm{mod}\ P)$ for which the n-th term of the series is divisible by p together with the number of such $n < P$. The primes p considered are those $\leqslant 31$.

702[G].—RAGY H. MAKAR, "The irreducible representation of the symmetric group of degrees 3, 4, and 5," Math. and Phys. Soc. of Egypt, *Proc.*, v. 3, 1948, p. 13–21.

The matrix representations of the elements of the symmetric groups of degrees 3, 4, and 5 are set forth in an abbreviated tabular form.

703[G].—JOHN RIORDAN. "Inversion formulas in normal variable mapping," *Annals Math. Stat.*, v. 20, 1949, p. 417–424.

If $G_1(g), G_2(g), \cdots$ are assigned polynomials, and if

$$x = g + \sum_{n=1}^{\infty} G_n(g)y^n/n!,$$

defines x in terms of g and a parameter y, then

$$g = x + \sum_{n=1}^{\infty} X_n(x)y^n/n!,$$

where

$$-X_n = Y_n(aG_1(x), aG_2(x), \cdots, aG_n(x)),$$

Y_n being the multivariate polynomial of the reviewer[1] in the variables $G_1(x)$ to $G_n(x)$ and the symbolic variable a which is such that

$$a^i \equiv a_i = (-d/dx)^{i-1},$$

with differentiations on all products $G_1(x)$ to $G_n(x)$ associated with it in the polynomial. This is the author's *first inversion formula*. Table 1 gives the explicit forms of $Y_n\ (fg_1, fg_2, \cdots, fg_n)$ for $n = 1(1)8$.

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[1] E. T. BELL, "Exponential polynomials," *Annals of Math.*, v. 35, 1934, p. 258–277.

704[I].—R. E. GREENWOOD & M. B. DANFORD, "Numerical integration with a weight function x," *Jn. Math. Phys.*, v. 28, 1949, p. 99–106.

The author considers two quadrature formulas

$$(1)\qquad \int_0^1 xf(x)dx = \frac{1}{2n}\sum_{i=1}^{n} f(x_{i,n}) + R_n^{(1)}(f)$$

$$(2)\qquad \int_{-1}^1 xf(x) = k_n \sum_{i=1}^{n} [f(y_{i,n}) - f(y_{i+n,n})] + R_n^{(2)}(f)$$

suggested by CHEBYSHEV[1] and of interest because the equal coefficients on the right minimize the probable error in the "observed" values of f.

As in the case of the unweighted formula

$$(3) \qquad \int_{-1}^{1} f(x)dx = \frac{2}{n}\sum_{i=1}^{n} f(x_{i,n})$$

of Chebyshev,[1] the optimal quantities $x_{i,n}$, $y_{i,n}$ are algebraic numbers which have, as n increases, the unpleasant tendency of leaving the interval of integration thus rendering the proposed quadrature useless [*MTAC*, v. 3, p. 97]. This phenomenon occurs in the case of (1) and (2) for $n \geqslant 4$, whereas in the case of (3) it occurs for $n = 8, 10, 11, \cdots$.

The present paper gives $x_{i,n}$ to 8D for $n = 1, 2, 3$, and k_n, $y_{i,n}$ for $n \leqslant 4$. For $n = 2, 3$ there are two possible values of k_n and two sets of y's; for $n = 4$, there are four values of k_n and four sets of y's. All results are to 8D, except for $n = 4$, when only 7D are given. The formula (2) for $n = 3$ is illustrated in two cases for $f(x) = e^x$ and compared with (3) for $n = 6$ and the NEWTON-COTES formula with 7 ordinates. The results speak well for (2).

D. H. L.

[1] P. L. CHEBYSHEV, "Sur les Quadratures," *Jn. de Math.*, s. 2, v. 19, 1874, p. 19–34. *Oeuvres*, St. Petersberg, v. 2, 1907, p. 165–180. See also R. RADAN, "Sur les formules de Quadrature a coefficients egaux," Inst. de France, Acad. Sci., *Comptes Rendus*, v. 90, 1890, p. 500–503, which contains data on (2) for $n = 2, 3$; a comparison with results of the present paper shows a number of minor errata.

705[I, M].—H. E. SALZER, "Coefficients for repeated integration with central differences," *Jn. Math. Phys.*, v. 28, 1949, p. 54–61.

In a previous paper[1] [*MTAC*, v. 3, p. 107] the author has given a table of coefficients for the repeated integration with forward and backward differences. In the present note the coefficients are based on central differences and were obtained by repeated integration of EVERETT's interpolation formula. The table extends from the case of 2-fold integration to 6-fold integration. In the important case of 2-fold integration the first 25 pairs of coefficients are given, the first 11 exactly and the others to 16D. For k-fold integration $k = 3(1)6$ the coefficients, which are all small, are given to 8 or 9S.

[1] H. E. SALZER, "Table of coefficients for repeated integration with differences," *Phil. Mag.*, s. 7, v. 38, 1947, p. 331–338.

706[K].—S. CHANDRASEKHAR, "On a class of probability distributions," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 219–224.

The function β_p under tabulation is the p-th moment of $W(\beta)$, where

$$W(\beta) = 2\beta\pi^{-1}\int_0^\infty e^{-u}y \sin \beta y dy, \quad u = y^{2/n}.$$

The function β_p is given explicitly by

$$\beta_p = 2\pi^{-1}(p+1)\Gamma(p)\Gamma(1 - np/3) \sin \tfrac{1}{2}\pi p$$

and is tabulated to 5S for

$$n = 1.6,\ p = .25(.25)1.75,\ 1.80,\ 1.85$$
$$n = 2,\ p = .25(.25)1.25(.05)1.45,\ 1.475$$
$$n = 3,\ p = .2(.2).8,\ .9,\ .95,\ .975$$
$$n = 4,\ p = .1(.1).5,\ .55,\ .575$$
$$n = 6,\ p = .1(.1).4,\ .45,\ .475$$
$$n = 8,\ p = .1(.1).3,\ .325,\ .35,\ .36$$
$$n = 10,\ p = .1,\ .2,\ .25,\ .275,\ .280$$

On p. 222, there is a 5S table of

$$[\tfrac{2}{3}\pi n(n+3)^{-1}\Gamma(3/n)\sin(\tfrac{2}{3}\pi n^{-1})]^{n/3}$$

for $n = 1.51, 1.52(.02)1.6(.1)2(.5)4(2)10(5)25$.

707[K].—H. J. GODWIN, "Some low moments of order statistics," *Annals of Math. Stat.*, v. 20, 1949, p. 279–285.

Let $\chi(i, n)$ be the i-th largest in a sample of n from the normal population with density function $(2\pi)^{-\frac{1}{2}}e^{-t^2/2}$. HASTINGS *et al.*[1] gave (a) the expectations and standard deviations of $\chi(i, n)$ to 5D, and (b) the covariances to 2D, for $n = 1(1)10$ and all i. JONES[2] obtained some of these values explicitly for $n = 4$. In the present paper more exact numerical integration is employed to improve the accuracy of tables in footnote 1, giving (a) to 7D and (b) to 5D. Correlations are given in a 4D table. The author also extends the results of Jones,[2] providing 26 new explicit values, for $4 \leqq n \leqq 6$.

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[1] CECIL HASTINGS, FREDERICK MOSTELLER, JOHN W. TUKEY, & CHARLES P. WINSOR, "Low moments for small samples: a comparative study of order statistics," *Annals of Math. Stat.*, v. 18, 1947, p. 413–426.

[2] HOWARD L. JONES, "Exact lower moments of order statistics in small samples from a normal distribution," *Annals of Math. Stat.*, v. 19, 1948, p. 270–273.

708[K].—FRANK E. GRUBBS, "On designing single sampling inspection plans," *Annals of Math. Stat.*, v. 20, 1949, p. 242–256.

Let $P(c, n, p) = \sum_{k=0}^{c} \binom{n}{k} p^k(1-p)^{n-k}$, and define p_1 and p_2 by $P(c, n, p_1) = 0.95$ and $P(c, n, p_2) = 0.1$. Interpolating to about 3S in published tables[1] [*MTAC*, v. 1, p. 76–79] of the beta and F distributions, the author tables p_1 and p_2 for $c = 0(1)9$, $n = 1(1)150$. A corresponding single entry table based on the POISSON approximation is also given. The tables are intended to aid in selecting sample size and acceptance number in sampling inspection plans having 5 per cent producer's risk and 10 per cent consumer's risk.

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[1] CATHERINE M. THOMPSON, "Tables of percentage points of the incomplete beta-function," *Biometrika*, v. 32, 1941, p. 168–181.

M. MERRINGTON & C. M. THOMPSON, "Tables of the percentage points of the inverted beta (F) distribution," *Biometrika*, v. 33, 1943, p. 73–88.

709[K, L].—E. J. GUMBEL, "Asymptotic distribution of range from that of reduced range," *Annals of Math. Stat.*, v. 18, 1947, p. 384–412.

The author considers the functions

$$\psi(x) = e^{-x} \int_{-\infty}^{\infty} \exp(-e^{t} - e^{-t-x})dt$$

and

$$\Psi(x) = \int_{-\infty}^{\infty} \exp(t - e^{t} - e^{-t-x})dt.$$

Table 1, p. 193–196, gives $\Psi(x)$, $\Delta\Psi$, $\psi(x)$ to 5D for $x = -3(.5)10.5$.

Table 1A, p. 397, gives the inverse function $\Psi^{-1}(x)$ to 2D for $\Psi(x) = .0002(.0001).001(.001).01(.01).1(.1).9(.01).99(.001).999(.0001).9997$.

710[L].—ANDRÉ ANGOT, *Compléments de Mathématiques à l'Usage des Ingénieurs de l'Électrotechnique et des Télécommunications.* Préface de Louis de Broglie. Paris, Éditions de la Revue d'Optique, 1949, viii, 660 p. 16.3 × 25.3 cm. Price 2500 francs, unbound.

This volume, by a professor at the École Supérieure d'Électricité and a "Lieutenant-Colonel des Transmissions," contains tables and rather rough graphs which may be noted even if the tables contain nothing new. In no case has the originator of any table been definitely indicated.

P. 333–334: Graphs of sinh$x$, cosh$x$, tanh$x$ and 5 or 6S tables of e^{x}, e^{-x}, cosh$x$, sinh$x$ for $x = 0(.2)6$.

P. 336–339: $\mathrm{si}(x) = -\int_x^\infty t^{-1} \sin t\, dt$, $\mathrm{Si}(x) = \frac{1}{2}\pi + \mathrm{si}(x)$. There are tables of Si(x), Ci(x) for $x = 0(.01)1(.1)6(1)15(5)100(10)200(100)1000$, 10^4, 10^5, 10^6, 10^7, 4 ∞ D; 4–7D, mostly 4D. There are also 5–6D tables of the maxima and minima of Ci(x) for $x/\pi = .5(.5)15.5$ and of Si(x) for $x/\pi = 1(1)15$, as well as graphs of Ci(x) and Si(x).

P. 342: Graphs and 5D table of $\theta(x) = 2\pi^{-\frac{1}{2}}\int_0^x e^{-t^2}dt$ with Δ for $x = .05(.05)2$.

P. 349–350: Graph and table of $\Gamma(1 + x)$, for $x = [0(.01)2;\ 4D]$, $x = [2(.01)3.99;\ 4\text{–}5S]$. Apparently reprinted from JAHNKE & EMDE.

P. 375–376: Graphs of $I_n(x)$, $n = 0(1)11$; $K_n(x)$, $n = 0, 1$; those of $I_n(x)$ apparently copied from Jahnke & Emde.

P. 380: Graphs of ber(x), bei(x), $M_0(x)$, $\theta_0(x)$.

P. 403–407: Tables of $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$ for $x = [0(.1)16;\ 4D]$.

P. 408–409: Tables to 4D of $J_n(x)$, $n = 2(1)9$, $x = 0(1)24$; $n = 10(1)17$, $x = 4(1)29$.

P. 410: First to ninth roots (4D) of $J_n'(x) = 0$, $n = 0(1)22$. Also first to eighth root (4–5D) of $J_n(x) = 0$, $n = 0(1)19$.

P. 411–412: Tables of $J_{n/2}(x)$, $\pm n = 1(2)13$, $x = [0(1)24;\ 4D]$.

P. 413–415: Tables of ber$x$, ber'x, bei$x$, bei'x, ker$x$, ker'x, kei$x$, kei'x for $x = 0(.1)10$ mostly 4S.

P. 416–417. Tables of $M_0(x)$, $\theta_0(x)$, $M_1(x)$, $\theta_1(x)$, $x = 0(.05)1.7(.1)3(.2)5(.5)6\text{-}(1)12(2)20(5)45$.

P. 442–444: Tables of LEGENDRE polynomials $P_n(x)$, $n = 1(1)7$, $x = [0(.01)1;\ 4D]$, apparently reprinted from Jahnke & Emde.

P. 445–446. Graphs of the associated Legendre functions of the first kind. Apparently taken from Jahnke & Emde, figs. 60–63.

P. 497–505: Tables of LAPLACE transforms; graphs of discontinuous functions, p. 502–505.

R. C. A.

711[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 12: *Tables of the Bessel Functions of the First Kind of Orders fifty-two through sixty-three.* Cambridge, Mass., Harvard University Press, 1949, x, 544 p., 20 × 26.6 cm. Offset print. Price $8.00.

Sixteen previously published volumes of the *Annals* have been reviewed in *MTAC*, v. 2, p. 176–177, 185–187, 261–262, 344, 368, v. 3, p. 41, 102, 185–186, 311–314, 367, 432–440, 474–475, 517–518. These volumes on publication listed at $10.00 each, are now listed at $8.00 each, which is more reasonable for offset-printed volumes.

The volume under review is the tenth in the Harvard series of tables of Bessel functions of the first kind, which, after three more volumes have been published, will contain tables, to 10D at least, for $J_n(x)$, for $n = 0(1)100$, and $x = 0(.01)100$; for $n = 0(1)3$, $x = [0(.001)25(.01)99.99;\ 18D]$, and $n = 4(1)15$, $x = [0(.001)25(.01)99.99;\ 10D]$, but beginning with order 16 the argument interval of the tables is constantly .01. The values of $J_n(100)$, $n = 0(1)100$ are to be given in the final volume 15. In the first two volumes only two orders were tabulated, while there are twelve in the current volume in which the first significant values .00000 00001 occur in connection with $J_{52}(27.53)$ and finally $J_{63}(36.34)$.

The tables in this volume are wholly new. The Harvard Computation Laboratory is at present the outstanding center in the world for the computation and publication of mathematical tables. In five years not only have there been 15 volumes of this kind in the *Annals* series, but also other tables which have been reviewed in *MTAC*, v. 2, p. 218, 300, 307.

R. C. A.

712[L].—MARIETTE LAURENT, "Table de la fonction elliptique de Dixon pour l'intervalle 0–0.1030," Acad. r. de. Belgique, *classe des sciences, Bull.*, s. 5, v. 35, 1949, p. 439–450, 15.8 × 25.1 cm.

On p. 441–445 is a table of values of $sm\, u$ for $u = [0(.001).103;\ 10D]$, Δ^2, where $x = sm\, u$ and $u = \int_0^x (1 - t^3)^{-2/3} dt$, and on p. 446–450 is a table of u with argument $sm\, u = [0(.001).103;\ 10D]$, Δ^2. The author states that 11 decimals were used in the calculations, the 9-th decimal corresponding to the precision of a centimeter in geodesic applications, and that there may be unit errors in the tenth decimal place.

For previous tables see *MTAC*, v. 3, p. 249, and A. C. DIXON, *Quart. Jn. Math.*, v. 24, 1890, p. 167–233. The connection between the Dixon function $sm\, u$ and the equianharmonic WEIERSTRASS function is given by $sm\, u = [2\sqrt{3}\wp(u/\sqrt{3})]/[\sqrt{3} - \wp'(u/\sqrt{3})]$.

R. C. A.

713[L].—A. D. MacDonald, "Properties of the confluent hypergeometric function," *Jn. Math. Phys.*, v. 28, 1949, p. 183–191, 17.4 × 25.3 cm.

On p. 184–190 are 6S tables for $z = .5(.5)8$, and $\alpha = .001, .01, .05, .1, .2, .25, .3(.1)1$ of

$$M(\alpha; \gamma; z) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\alpha + n)z^n}{\Gamma(\alpha)\Gamma(\gamma+n)n!}$$

for $\gamma = \frac{1}{2}(\frac{1}{2})2$, and of the logarithmic solution for $\gamma = 1, 2, 3$.

R. C. A.

714[L].—Z. Mursi, "On the relation of Airy and allied integrals to the Bessel functions," Math. and Phys. Soc. of Egypt, *Proc.*, v. 3, 1948, p. 23–38.

If $f(x)$ stands for one of the functions $Ai(x)$, $Bi(x)$ or $aAi(x) + bBi(x)$, then the derivatives of $f(x)$ can be expressed as

$$f^{(2n)}(x) = P_n f(x) + Q_n f'(x)$$

$$f^{(2n+1)}(x) = R_n f(x) + S_n f'(x),$$

where P_n, Q_n, R_n, and S_n are polynomials in x. These polynomials are tabulated on p. 37–38 for $n = 1(1)15$.

R. C. A.

715[L].—Fritz Oberhettinger & Wilhelm Magnus, *Anwendung der elliptischen Functionen in Physik and Technik.* (*Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*, v. 55.) Berlin-Göttingen-Heidelberg, Springer, 1949, viii, 126 p. 16 × 24.1 cm. Price 15.6 German marks in paper cover; bound 18.3.

Tables on p. 43–126 are as follows:

A. Complete normal elliptic integrals.

(i) 5D values of $k^2 = \sin^2 \alpha$, K and E for $\alpha = 0(1°)70°(30')80°(12')89°(6')90°$.

(ii) 5D values of K, K', K'/K, K/K', $\log q$, $\log q'$ for $k^2 = 0(.01).5$.

(iii) 5D values of K, K', K'/K, K/K' for $k^2 = .000001(.000001).00001, .0001(.0001).003$.

B. Tables of normal elliptic integrals of the first kind; values of $F(k, \phi)$, $k = \sin \alpha$ for $\alpha = 5°(5°)90°$ and $\phi = [1°(1°)90°$; 5D] values for $\phi \leqslant 5°$, 4D values for $\phi > 5°$.

C. Tables of normal elliptic integrals of the second kind, 4D values of $E(k, \phi)$, $k = \sin \alpha$, for $\alpha = 5°(5°)90°$ and $\phi = 1°(1°)90°$.

R. C. A.

716[L].—W. Peremans & J. Kemperman, "Nummeringspribleem van S. Dockx, Mathematisch Centrum. Amsterdam," *Rapport* ZW; 1949–005,4 leaves, 19.8 × 34 cm.

On leaf 4 is a table of $a_k = \frac{1}{6}k(k + 1)(2k + 1) = \frac{1}{3}B_3(k + 1)$, where B_3 is the third Bernoulli polynomial, for $k = 1(1)100$.

R. C. A.

717[L].—R. A. Rankin, "The theory of the motion of rotated and unrotated rockets," R. Soc. London, *Phil. Trans.*, v. 241, 1949, p. 457–585, 23.4 × 29.9 cm.

This work contains two tables of "Fresnel functions." The main table gives

$$A(x) = \pi^{-1}2^{-\frac{1}{2}} \int_0^\infty t^{-\frac{1}{2}}(1+t^2)^{-1} \exp\left(-\tfrac{1}{2}\pi x^2 t\right)dt$$
$$= \{\tfrac{1}{2} - S(x)\} \cos \tfrac{1}{2}\pi x^2 - \{\tfrac{1}{2} - C(x)\} \sin \tfrac{1}{2}\pi x^2$$

and

$$B(x) = \pi^{-1}2^{-\frac{1}{2}} \int_0^\infty t^{\frac{1}{2}}(1+t^2)^{-1} \exp\left(-\tfrac{1}{2}\pi x^2 t\right)dt$$
$$= \{\tfrac{1}{2} - S(x)\} \sin \tfrac{1}{2}\pi x^2 + \{\tfrac{1}{2} - C(x)\} \cos \tfrac{1}{2}\pi x^2$$

and

$$Z(x) = \pi \int_0^x A(u)du,$$

where $S(x)$ and $C(x)$ are the Fresnel integrals

$$C(x) = \int_0^x \cos \tfrac{1}{2}\pi u^2 du, \quad S(x) = \int_0^x \sin \tfrac{1}{2}\pi u^2 du.$$

These are tabulated to 4D for $x = 0(.01)1(.05)1.5(.1)7.(.2)10(.5)15$ with first differences, and in the case of $Z(x)$, second differences. For $x \geq 1$ the table gives also

$$A_1(x) = (\pi x)^{-1} - A(x)$$
$$Z_1(x) = Z(x) - \ln x.$$

The second table gives 4D values of the integrals

$$A^*(x) = \int_x^\infty A(u)du/u$$

$$B^*(x) = \int_x^\infty B(u)du/u$$

for $x = 0(.1)5$ with first and second differences. For $x < 1$, $A^*(x) + \ln x$ and $B^*(x) + \ln x$ are also given.

D. H. L.

718[L].—Norbert Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications.* Published jointly by Mass. Inst. Technology, Cambridge, Mass., and John Wiley & Sons, New York, 1949, x, 163 p. 14.6 × 22.6 cm. Price $4.00.

Appendix A of the work consists of a table of what the author calls Laguerre Functions. These are more explicitly

$$F_n(x) = (-1)^n 2^{\frac{1}{2}} e^{-x} L_n(2x)/n!$$

where $L_n(t)$ is the usual Laguerre polynomial given by

$$L_n(t) = e^t d^n(t^n e^{-t})/dt^n = n!M(-n, 1, t),$$

M being the confluent hypergeometric function. Thus

$$L_4(t) = t^4 - 16t^3 + 72t^2 - 96t + 24.$$

The range of x and n is

$$x = 0(.01).1(.1)18(.2)20(.5)21(1)26(2)30; \quad n = 0(1)5.$$

The table is given mostly to 5S, but in some cases only to 3S. A spot check reveals a number of last digit errors and the following gross error

$$x = 4.7, n = 4, \quad \textit{for } 88260 \quad \textit{read } 89267.$$

No statement is made as to the method of construction of the table. Apparently no really good tables of $L_n(t)$ have been published. See *MTAC*, v. 2, p. 267, FMR *Index*, p. 337.

D. H. L.

719[Q].—V. KRAT & S. PETROV, "Tablit͡sy vspomogatelnich funkt͡sii ψ i χ dli͡a opredeleni͡a elementov sistem zatmennykh peremenykh" (Tables of auxiliary functions ψ and χ for determination of the elements of systems of eclipsing variables) II, Central Astronomical Observatory, Pulkovo, *Izvestia*, v. 17, No. 5, 1947, p. 117.

This paper consists, in essence, of three tables. Tables 1 and 2 contain numerical values of RUSSELL's well-known $\psi(k, \alpha = n)$ function which is needed for the computation of the ratio k of the radii of components of an eclipsing binary system from an analysis of light curves due to total eclipses of a star which is completely darkened at the limb. A definition of this auxiliary function in terms of the basic p-functions was first given by Russell (*Astrophys. Jn.*, v. 35, 1912, p. 315); it is repeated in the introduction to the tables under review.

Of these, Table 1 contains 3D values of $\psi(k, n)$ appropriate for the partial phase of a total eclipse, while Table 2 gives values of the same function appropriate for the annular phase of a transit. The arguments of tabulation are $k = .1(.1)1$, $n = 0(.1)1$ for Table 1, and $k = 2(.1)1$, $n = 0(.1)1$ for Table 2. The intervals of tabulation in both arguments are too large to make the tables easy of interpolation. Both tables are not original, but are revised versions of earlier tables of the same functions published by RUSSELL & SHAPLEY (*Astrophys. Jn.*, v. 36, 1912, p. 239; Table IIx on p. 245 corresponds to Krat and Petrov's Table 1; while Table IIy on p. 391 of the same volume of the *Astrophys. Jn.* corresponds to Krat and Petrov's Table 2). A comparison of the corresponding entries of the new and old tables reveals discrepancies attaining the second significant place, and due no doubt to the inferior accuracy of the old tables which were based on inaccurate p-functions. The new Russian tables are based on the extensive and accurate 5D tables of $p(k, n)$ which were published in 1939 by T͡SESEVICH [*MTAC*, v. 3, p. 191–194].

Table 3—the main feature of the paper under review—contains a set of 4D tables of Krat's auxiliary functions $\psi(k, n\alpha_0)$ and $\chi(k, \alpha_0)$ appropriate for partial eclipses of stars exhibiting uniformly bright disks. The arguments of tabulation are $k = .1(.1)1$, $\alpha_0 = .1(.1)1$, and $n = 0(.1).9$. The reader is cautioned to notice that Krat's functions ψ and χ are *not* identical with

Russell's well-known functions denoted currently by the same symbols; for the definition of Krat's functions tabulated in the paper under review cf. *Russian Astronomical Jn.*, v. 11, 1934, p. 412 (Russian, with English summary).

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720[R].—KARL REICHENEDER, "Fehlertheorie und Ausgleichung von Rautenketten in der Nadirtriangulation," Deutsche Akademie der Wissenschaften zu Berlin, *Veroffentlichungen des Geodätischen Institutes in Potsdam*, No. 1, 1949, viii, 98 p., 20.7 × 29.6 cm.

Contains tables of the probable errors in photogrammetric control extension by nadir point triangulation. The tables, p. 82–96, are based on a triangulation net of fifteen rhombuses (nadir numbers, N_0 to N_{15}), in which are located two fixed points. Fifteen error-tables are given, in which N_0 is the first fixed point, and the second fixed point is placed successively at $N_1, N_2, \cdots$ to N_{15}.

On p. 28 is a table giving the coefficients in the expansion of Lucas $U_n = (a^n - b^n)/(a - b)$ and $V_n = a^n + b^n$ as polynomials in $a + b$ and ab for $n = 0(1)16$ together with numerical values of U_n and V_n in the cases $ab = 1, a + b = 3, 4$. These are used to compute tables of weighting coefficients p. 34–46.

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721[V].—E. N. FOX, "The diffraction of two dimensional sound pulses incident on an infinite uniform slit in a perfectly reflecting screen," R. Soc. London, *Phil. Trans.*, v. 242, 1949, p. 1–32.

On p. 19 there are two tables of the function $G_0(Y, \tau) = \dfrac{2}{\pi} \tan^{-1} (\tau/Y)^{\frac{1}{2}}$ for $\tau = 0(.2)3.8$, $Y = 2.2(.2)3.8$ and for $\tau = 0(.2)1.8$, $Y = 2.2(.2)3.8$.

On p. 20, there are tables of

$$f_1(Y, \tau) = G_0(Y, \tau + 1) - G_0(Y, 1) - \frac{1}{\pi}\int_0^{\tau/2} \frac{Y^{\frac{1}{2}}G_0(x, \tau - 2x)}{(1 + x)^{\frac{1}{2}}(1 + x + Y)}\, dx$$

for $\tau = 0(.2)2.8$, $Y = .1(.1)1.0$ and for $\tau = 0(.2)1.8$, $Y = 1.2(.2)3.0$ and a table of

$$f_2(Y, \tau) = \frac{1}{\pi}\int_0^{\tau/2} \frac{Y^{\frac{1}{2}}f_1(x, \tau - 2x)}{(1 + x)^{\frac{1}{2}}(1 + x + Y)}\, dx$$

for $\tau = 0(.2)1.8$, $Y = .2(.2)2.0$. All tables are to 4D.

722[V].—VIRGINIA GRIFFING & FRANCIS E. FOX, "Theory of ultrasonic intensity gain due to concave reflectors," Acoustical Soc. of Amer., *Jn.*, v. 21, 1949, p. 348–359.

On p. 350 there is a table of $[Si(k) - (1 - \cos k)/k]/\pi$ for $k = .1(.1)$-$.4(.2)2(.5)5(1)16, 20(10)50$, and for $k = n\pi$ with $n = 1(1)5$ to 4S.

MATHEMATICAL TABLES—ERRATA

References to Errata have been made in RMT **693** (Mather), **704** (Greenwood & Danford), **718** (Wiener), **719** (Krat & Petrov).

162.—R. A. FISHER & F. YATES, *Statistical Tables for Biological and Medical Research*, Edinburgh, 1st ed. 1938, 2nd ed. 1943, 3rd ed. 1948.

Table 22, Initial differences of powers of natural numbers

r	*s*	*for*	*read*
12	19	51330	51300
12	21	31078	30178
12	22	61937	51137
12	23	27736 13530	27734 83930
12	24	45923 13460	45907 58260
12	25	08035 37080	07848 74680
13	23	52190	41390
13	24	60581 92000	60579 22000
13	25	33488 09460	33437 44260
14	24	51800	41000
14	25	03613 17200	03608 96000
15	25	58000	47200

These errata are in all three editions.

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Table 17, 3rd ed. only, Solution no. 11 should read

"Use 19, taking any set of varieties occurring in the same block, and deleting that block."

Professor W. L. STEVENS has recently derived a cyclic solution in two families for this design, namely:

$$\begin{matrix} a & b & c & e \\ a & d & f & i \end{matrix}$$

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163.—FMR *Index*, p. 39, line −7, reference to actual partitions $n = 1(1)18$, CAYLEY 1881. In author's Index, p. 385, Cayley, 1881, refers incorrectly to *Trans., Camb. Phil. Soc.*, 13 and *Coll. Math. Papers*, 11, 144–147. In fact the correct reference is given in LEHMER's Guide, p. 90, CAYLEY 4 to *Am. Jn. Math.*, v. 4, 1881, p. 248–255 and *Coll. Math. Papers*, v. 11, 1896, p. 357–364.

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164.—D. H. LEHMER, "Note on an absolute constant of Khintchine," *Amer. Math. Monthly*, v. 46, 1939, p. 148–152.

The constant K mentioned in the title is the limit, as $n \to \infty$ of the geometric mean of the first n partial quotients in the continued fraction expansion of almost all real numbers. KHINTCHINE[1] has shown that this limit exists and its logarithm is given by:

$$\ln 2 \ln K = \sum_{r=2}^{\infty} \ln r \ln (1 + (r(r+2))^{-1}) = S.$$

Khintchine gave $K = 2.6$. Lehmer found that $S = .6847248$ but, in passing from S to K, erroneously concluded that $K = 2.685550$. This value is quoted in the FMR *Index*, p. 107. A transformation of the slowly converging series for S gives

$$S = \ln 2 - \tfrac{1}{2} \sum_{k=2}^{\infty} k^{-1}(1 - \zeta(2k)) \sum_{r=1}^{k-1} r^{-1}(2r+1)^{-1},$$

where ζ is RIEMANN's function. From tables[2] of this function we find that

$$S = .68472478856$$

and hence

$$K = 2.685452001.$$

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[1] A. KHINTCHINE, "Zur metrische Kettenbruchtheorie," *Compositio Math.*, v. 3, 1936, p. 276–285.

[2] H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 2, Bloomington, 1935, p. 244.

165.—NBSMTP, *Tables of Fractional Powers*, Columbia University Press, N. Y., 1946. There is a last figure error in the 15D value of π^{10} in table 3, p. 34. In fact the correct value to 20D is given by J. T. PETERS, *Zehnstellige Logarithmentafel. Erster Band*, . . . *Anhang* by PETERS & STEIN, Berlin, 1922.

$$\pi^{10} = 93648.04747\ 60830\ 20973\ 71669.$$

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166.—KEIKITIRO TANI, *Tables of si(x) and ci(x) for the Range x = 0 to x = 50*. Meguro, Tokyo, Naval Research and Experiment Establishment, 1931, iv, 128 p. 18.4 × 25.6 cm.

The following errors were found when the NBSCL was preparing its three volumes for tables of Si(x), Ci(x), Ei(x), $-$Ei($-x$), 1940–1942. In the Tani tables are values of $si(x)$ for $x = [0(.01)50; 6D]$, Δ; and values of $ci(x)$ for $x = [0(.0001).05(.001)1(.01)50; 6D]$, Δ.

Argument	*For* $ci(x)$	*Read* $ci(x)$	Argument	*For* $ci(x)$	*Read* $ci(x)$
0.057	−2.288200	−2.288300	2.34	0.335436	0.335434
0.512	−0.157037	−0.157039	2.38	0.326406	0.323405
0.656	+0.049348	+0.049948	2.39	0.320358	0.320356
0.843	0.233949	0.233943	2.45	0.301748	0.301746
1.67	0.468998	0.468996	3.91	−0.125351	−0.125349
1.68	0.468377	0.468375			
1.69	0.467701	0.467699	8.55	+0.095875	+0.095785

The argument following 2.58 *should read* 2.59, *not* 2.69; the argument following 5.74 *should read* 5.75, *not* 3.75.

Apart from the 13 errors greater than one unit in the last place, listed above, there were 334 last-place unit errors in values of $ci(x)$, and 127 such errors in $si(x)$.

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167.—J. TRAVERS, "Perfect numbers," *Math. Gazette*, v. 23, 1939, p. 302.

The 10-th and 11-th perfect numbers are given incorrectly. The 30-th digit (from the left) of P_{10} should be 4, not 2. The 27-th digit of P_{11} is missing; it should be 4.

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UNPUBLISHED MATHEMATICAL TABLES

EDITORIAL NOTE: Beginning with this issue we are starting a collection of unpublished mathematical tables to be known as the UMT FILE. Authors of tables which have no immediate prospect of publication are invited to submit copies for deposit in UMT FILE. Description of such tables will appear in UMT and photostat or microphilm copies will be supplied at cost to any reader of *MTAC*. Address tables or correspondence to D. H. LEHMER, 942 HILLDALE AVE., BERKELEY 8, CALIFORNIA.

84[C].—G. W. REITWIESNER, *Arccot 5 and arccot 239 to 2035 places*. On deposit in UMT FILE.

This is a by-product of the author's calculation of π [*MTAC*, v. 4, p. xx].

85[F].—D. JARDAN [YARDEN] & A. KATZ. *Additional page 477 to D. N. Lehmer's Factor Table*. On deposit in UMT FILE.

This single page is of the same form as Lehmer's well known factor table [Carnegie Institution of Washington, *Publ.* 105, 476 p.] and covers the range 10 017 000 − 10 038 000.

86[F].—SYLVESTER WHITTEN. *Tables of the totient and reduced totient function*. Manuscript deposited in UMT FILE.

The totient function, $\phi(n)$, is the number of numbers not exceeding n and relatively prime to n. The function $\phi(n)$ has the property that $a^{\phi(n)} - 1$ is divisible by n for all a prime to n. The reduced totient $\psi(n)$ is defined as the least positive number m such that $a^m - 1$ is divisible by n for all a prime to n so that $\psi(n) \leqslant \phi(n)$.

Table 1 gives $\phi(n)$, $\psi(n)$, and $\Phi(n) = \phi(1) + \phi(2) + \cdots + \phi(n)$ for $n = 1(1)1200$, and the approximation $3n^2\pi^{-2}$ to $\Phi(n)$ for $n = 1000(1)1200$ to
$(1064)^2$ 2D. The author notes that $\Phi(1063) = 344116 > 3(1063)^2\pi^{-2} = 344115.92$, contrary to a conjecture of J. J. SYLVESTER.

Table 2 gives $s(n) = \frac{1}{2}n\phi(n)$, the sum of the numbers $\leqslant n$ and prime to n. Also $S(n) = s(1) + s(2) + \cdots + s(n)$, and the approximation $n^3\pi^{-2}$ to $S(n)$ to 2D for $n = 1(1)100$.

EDITORIAL NOTE. The late Mr. Whitten, a telephone engineer, became interested in the above functions in connection with the problem of splicing telephone cables to minimize cross-talk. [See H. P. LAWTHER, "An application of number theory to the splicing of telephone cables," *Am. Math. Monthly*, v. 42, 1935, p. 81–91]. The above tables extend those of J. J. Sylvester and A. CAUCHY, see *Guide to Tables in the Theory of Numbers*, National Research Council, *Bulletin*, no. 105, 1941, p. 6, 7. [*MTAC*, v. 3, p. 531].

87[G, I].—H. E. SALZER, *Derivatives* of $[y(x)]^n$, Manuscript in the possession of the author.

This manuscript gives the explicit expressions as polynomials in y and its derivatives for the first twelve derivatives of $[y(x)]^n$, $n = 2(1)20$. The expression $d^m[y(x)]^n/dx^m$ is a sum of m terms, each term consisting of two factors, one factor depending only on n, the other factor depending only on m. The manuscript is in the form of two separate superimposable tables, which enables the user to place in juxtaposition the two factors corresponding to each of the m terms of $d^m[y(x)]^n/dx^m$.

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88[L].—R. T. BIRGE, *Table of Fresnel Integrals*, 3 mimeographed sheets, Dept. of Physics, University of California, Berkeley. Copy in UMT FILE.

The table gives the Fresnel integrals

$$C(u) = \int_0^u \cos\left(\tfrac{1}{2}\pi x^2\right)dx \qquad \text{and} \qquad S(u) = \int_0^u \sin\left(\tfrac{1}{2}\pi x^2\right)dx$$

for $u = [0(.05)12.05; 4D]$. These tables extend those of C. M. SPARROW beyond $u = 8$. For reference to these and other tables of Fresnel integrals see *MTAC*, v. 3, p. 479, v. 4, p. 58.

89[L].—CARL HAMMER, *Systems of particular solutions of the differential equation* $y^{(n+1)} + y = 0$, *and numerical tables*, manuscript in the possession of the author, Walter Harvey Junior College, N. Y., and of the Library at Brown University, 6 sheets.

The functions tabulated are

$$H_{nk}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{(n+1)r+k}}{[(n+1)r+k]!} \quad (k = 0, 1, 2, \cdots, n).$$

These solutions are given explicitly in terms of trigonometric and hyperbolic functions for $n = 0(1)5$. Actual values of the functions are given to 5D for $n = 2(1)5$ and for $x = -5(.1)5$.

AUTOMATIC COMPUTING MACHINERY

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Technical Developments

Magnetic Drum Storage for Digital Information Processing Systems

Introduction.—Automatic digital computers belong to a class of devices which may be described by the term "information processing systems." Some further examples of information processing systems (hereinafter abbreviated IPS) are statistical analysis machines, airline reservation tallying systems, airport traffic control systems, and inventory record systems. A requisite component of every such device is a storage section which serves as a repository for a number of separate items of information. These items, which we shall call "words," may be numerical quantities, alphabetical material, machine instruction codes, or combinations of these.

In many applications it is required to be able to refer at random to any word in storage. Each position in storage which may be occupied by a word is therefore designated by a number called an "address." This type of information store is analogous to a function table, since to each value of the argument, or address, there corresponds a single function value, or stored quantity. In such systems it is necessary to be able to read the word stored at a given position an indefinite number of times without deterioration of that word or of its neighbors. It is also generally necessary to be able to replace or alter the word at a given storage position without disturbing the contents of neighboring positions. Alterable information storage may be physically realized in a number of different ways. Well-known examples of these are electromechanical relays and stepping switches, acoustic delay lines, electrostatic storage tubes, and various magnetic recording devices.

The choice of storage media to suit a given application is governed by several considerations. One of these is the degree of physical stability required of the stored data. In certain applications it is necessary to retain stored information for extended periods, perhaps for weeks at a time. Under such circumstances it is a distinct advantage if the retention of data in storage does not depend on or require the continued operation of electric circuits. For if the stored information is not "volatile," it is possible to shut down the equipment for overnight periods, or for the purpose of maintenance, without loss of data. The *combined* properties of alterability and non-volatility are exhibited by very few of the known physical means of storage. Among these few are stepping switches, latching relays and signals recorded on magnetizable media.

A second consideration governing the choice of storage media is the required capacity, a quantity usually expressed as the number of binary digits or bits of information to be stored. Where large storage is required, the bulky relays and stepping switches present a serious space problem. A two-position relay is capable of storing only a single binary digit of information, while an n-position stepping switch stores only $\log_2 n$ binary digits of information.

With magnetically recorded signals, on the other hand, a large quantity of information may be stored in relatively compact form, as will be shown.

A third consideration is the permissible "access time" or maximum waiting time which may be tolerated in searching for a given storage position and reading or altering the word stored there. Many applications require quick access to any position in the storage. One practical way to satisfy the need for quick random access is to scan the entire mass of stored data continuously at a rapid repetition rate. In the storage system to be described information coded in terms of the binary digits 1 and 0 is recorded on a magnetizable medium on the surface of a continuously rotating cylindrical drum. The small magnetized areas corresponding to individual digits are arranged in parallel peripheral tracks about the drum. Near each track is a single stationary magnetic head for reading and writing the digits in that track. Once in every revolution every magnetized area on the drum is thereby accessible for the effectively instantaneous operations of reading or writing. The maximum access time in this instance is equal to the rotation period of the drum.

The information is recorded in binary coded form so that it is necessary to distinguish between only two magnetic states for each elemental area. There is no need for over-all linearity of the recording and reproducing processes. Specifically, these two magnetic states correspond to positive and negative magnetization of the medium in a direction parallel to its motion. The binary coding requirement imposes no limitation on what may be stored, since information of any kind is readily expressed in a "1 − 0" or "on-off" code. Thus, decimal digits may be recorded as 4-digit binary code groups, and alphabetical characters may be recorded as 5- or 6-digit binary groups. This technique is commonly used in telegraphic systems and in electrical computing devices.

It is the purpose of the present paper to describe a practical method for the alterable, nonvolatile storage of information. For applications requiring these properties, the magnetic drum storage system provides what is felt to be a reasonable balance of the factors: access time, storage capacity, and bulk and cost of equipment.

Utilization of the Storage Surface.—Each word appears on the drum surface in parallel rather than in serial fashion. That is, each binary digit of a 30-digit word, for example, is represented by a single elemental magnetized area, or "cell," in each of 30 separate tracks, rather than by 30 cells in one track. As the drum rotates, the 30 digital cells representing one word pass simultaneously under the magnetic heads in their respective tracks. If each track should contain 4000 digital cells around its circumference, then a group of 30 tracks would store 4000 30-digit words. Several such groups of tracks may be needed to provide the required storage capacity.

The number of elemental areas per track and the number of groups of tracks on a drum are determined by the storage requirements of the application. Suppose that it is desired to store W words of b binary digits each, with maximum access time of T milliseconds. Let R represent the standard scanning rate, i.e., the number of digital cells passing a given magnetic head in a millisecond. Since there is a single read-write station per track, the drum rotation period may be made equal to T. Each head then scans RT digital cells in a revolution. In other words, each track stores RT binary digits. A

group of b tracks stores RT words. To provide storage for W words, W/RT such groups of tracks are needed.

For economy of equipment and space it is desirable that the number of binary digits of stored information under the control of each magnetic head be made as large as practicable. Since each track stores RT digits, this calls for a large value of the scanning rate, R. The scanning rate is equal to the product of the drum surface velocity and the number of digital cells per unit length of track. The values at which these quantities have been standardized are 1600 inches per second and 80 digital cells per inch, respectively, corresponding to a value of 128 digital cells per millisecond for R. These are conservative design constants which have been found entirely adequate for reliable discernment of the value of every stored binary digit.

There are eight tracks per axial inch along the drum. Since 80 binary digits are stored in each peripheral inch of track, the storage capacity of the drum surface is 640 digits per square inch. In other words, each binary digit is allocated a rectangular zone having effective dimensions of 0.125 inch parallel to the drum axis and 0.0125 inch peripherally, or perpendicular to the drum axis. This rectangular zone constitutes a digital cell. Whether the stored digit is a 1 or a 0 is established by the magnetic orientation or polarity of a slightly smaller region within this zone. The magnetic polarity of the surrounding area corresponds to the convention chosen to represent 0. A plot of the magnetic intensity along the center of a peripheral track will disclose regions oriented positively and negatively in a direction parallel to the track. If the positive polarity represents 1 and the negative polarity 0, a series of 0's would be characterized by uniform magnetization along the track. A series of 1's, on the other hand, would show up as a series of spots of positive polarity separated by small regions of negative polarity.

Magnetic Heads and Drum Surface Coating.—The magnetic head is a specially designed form of electromagnet with an elongated ring-shaped core. The core has a fine gap on the side adjacent to the drum surface. To write, a winding on the core is energized with a brief pulse of current. The minute area of drum surface under the gap at that instant is magnetized in a direction determined by the polarity of the current and the sense of the winding. The same head serves for reading. As successive digital cells pass under the gap, characteristic signal voltages are induced in a pickup winding on the core. These signals have amplitudes on the order of tenths of a volt. The reading operation does not disturb the stored data in any way.

Although the tracks are spaced eight to the inch along the drum, each magnetic head in its mounting assembly occupies a circular area approximately one inch in diameter, projected on the drum surface. For this reason all of the heads are not placed in a single line parallel to the drum axis but are staggered in position.

The magnetizable medium on the drum surface is a smooth, sprayed-on coating of magnetic iron oxide, the same material which is used on magnetic sound recording tape. This surface is protected by a thin over-coating of a hard lacquer.

Each digital cell passes under its magnetic head many times per second at a 90 mile per hour relative speed. These conditions preclude the use of the contact technique commonly employed in recording on magnetic tape. A clearance of 0.002 inch is therefore maintained between the magnetic

head and the drum surface. This noncontact clearance is the principal factor limiting the number of reliably resolvable digital cells per inch of track.

Functional Description of System.—The functional block diagram of a magnetic drum storage system is shown in Figure 1. The dotted boundary surrounds those units which would be considered part of the storage section of an IPS. The channels by which the storage section communicates with other sections of the IPS are shown along the lower edge of the boundary.

The external functions of the storage section are simple. If a word is to be written into storage, the information which must be transmitted to the storage section consists of: (1) the address of the desired storage position; (2) the word to be written; and (3) a control signal specifying that the operation is to "write." If the word occupying a given storage position is to be read, the required information consists of: (1) the address of the desired position; and (2) a control signal specifying that the word at that position is to be "read" to one of several possible destinations (buses to two destinations are shown).

The units with which the storage section communicates are determined by the nature of the IPS. In a computer, for example, the address and the control signals originate in the central program control of the computer. The word to be written may come from the arithmetic section. The destinations for words read out of storage may be the program control section, the arithmetic section, or a printing device.

The storage section communicates internally and externally on a parallel channel basis, in that the several binary digits of a word are transmitted at one time over as many electrical channels. Heavy lines in Figure 1 represent multi-channel buses for the transmission of words or addresses. Light lines represent single or multiple channels for control information. In each external channel, the presence of a pulse indicates a 1 and the absence of a pulse, a 0. The channels within the storage section carry more specialized forms of signals, such as d-c potentials, for example.

The word to be written and the address of the desired storage position are held, until completion of the operation, in the Insertion Register and the Address Register, respectively. These registers consist of toggle-circuits, or static flip-flops. A toggle-circuit is an electron tube circuit having two symmetrical stable states so that it is capable of holding a single binary digit of information.

Upon completion of the specified writing or reading operation, a control signal announcing completion is sent out by the storage section. At the same time, the Address and Insertion Registers are cleared, so that the storage section is then receptive to further assignment.

In the interest of clarity, the operation of the system will be described in terms of an example having a specific set of storage characteristics: (1) word size, b: 30 binary digits; (2) capacity in words, W: 8192 or 2^{13}; and (3) maximum access time, T: 16 milliseconds. The number of words which can be stored in each group of 30 tracks is equal to RT, or 2048 (128 times 16). Four track groups must therefore be provided, plus several additional tracks for location and timing purposes. These are indicated in Figure 1.

The 8192 storage positions are designated by 8192 addresses. While the addresses may be *any* set of 8192 distinct binary coded designations, that

set which consists of all the 13-digit binary numbers is the least redundant and most economical of equipment.

The 13-digit address is composed of two parts, a 2-digit "group index" and an 11-digit "angular index." The group index specifies one of the four, or 2^2, groups of tracks. The angular index specifies one of 2048, or 2^{11}, angular positions of the drum.

In addition to the 120 storage tracks, there are 11 angular index tracks and one timing track. These tracks contain permanently recorded information. The angular index tracks contain the 2048 11-digit angular indices. The timing track serves as a source of timing pulses, for precisely marking the instant at which the drum passes through each of its 2048 discrete angular positions. One of these timing pulses, selected on the basis of the desired angular index, denotes the instant at which the desired storage position is available for reading or writing.

Time-selection is performed by an 11-fold coincidence detector which continuously compares the desired 11-digit angular index in the Address Register with the outputs of the circuits which read the angular index tracks. As long as the scanned angular indices do not match the desired angular index, timing pulses cannot get through the coincidence detector. When the drum passes through the angular position at which a match occurs, a single time pulse is delivered to the storage control circuits for triggering of the appropriate writing or reading operation.

The function of the Writing Circuits is to replace the word at the specified storage position with the word standing in the Insertion Register. There is a Writing Circuit associated with each of the 120 storage track magnetic heads. A Writing Circuit contains two miniature thyratrons, each of which can discharge a simple network through a winding on the magnetic head. The 30 pairs of thyratrons in the selected group are simultaneously triggered by a pulse from the storage control circuits, but only one thyratron in each pair fires, one to write a 1, the other to write a 0. One of the thyratrons is prevented from firing by application of a negative bias to its shield grid. The choice of 1 or 0 is determined by the value of the corresponding digit in the Insertion Register.

It should be noted that there is no need for "erasure," as such, since the operation of writing a word into a storage position substitutes the new word for the previous contents of that position. The word stored at a given position is simply the one that was written there last.

The use of thyratrons instead of "hard" vacuum tubes in the Writing Circuits effects a significant saving in the number of tubes. The time which must elapse between successive writing operations is admittedly longer for thyratrons. However, the duty-cycle required of the writing operation is generally so low that this limitation is of little consequence.

The reading operation consists in transmitting to the specified destination the word stored at the position specified by the address in the Address Register. The units which participate in reading are indicated in Figure 1 as Reading Gates, Reading Circuits, and Output Gates.

Track group selection is accomplished in the Reading Gates. These are preamplifiers which are either blocked or operative, as determined by control voltages. Each of the 120 Reading Gates receives signals from its associated

magnetic head, but only the selected group of gates transmits signals to the Reading Circuits.

The Reading Circuits are 30 in number and consist of amplifiers and wave-form shaping circuits. These operate continuously on the signals originating in the selected group of tracks.

The amplified signals from the Reading Circuits are impressed on two sets of Output Gates, one for each destination. At the instant denoted by the selected time pulse, the appropriate set of Output Gates is pulsed. This operation, by sampling the signal stream from the Reading Circuits at the correct time, transmits the desired word to the specified destination.

The Storage Control Circuits consist of electronic switching and gating circuits for translating the group index code, the selected time pulse, and the external control signals into the appropriate group, time, and destination signals. The Storage Control Circuits also include automatic lockout delays which prevent a storage reference operation from following a previous one too closely to permit complete circuit recovery. These delays are of the order of 50 microseconds, except in the special case of a writing operation which follows a previous writing operation. In this case, the second writing operation must not take place until about 2 milliseconds after the first. If a storage reference operation is initiated too soon after a previous one and the desired angular index comes up before the lockout delays have cleared, the effect is simply to delay execution of the operation for one drum revolution.

TABLE I

Characteristics of 36 Magnetic Drum Storage Systems

CAPACITY (Wb), BINARY DIGITS	MAX ACCESS TIME (T) MILLISECONDS	DRUM DIMENSIONS, INCHES		MAGNETIC HEADS	WORD SIZE, b = 15 BINARY DIGITS				WORD SIZE, b = 30 BINARY DIGITS				WORD SIZE, b = 60 BINARY DIGITS			
		DIAMETER	LENGTH		WORDS (W)	TRACK GROUPS (W/128T)	ELECTRON TUBES	TUBES PER 1000 DIGITS	WORDS	TRACK GROUPS	ELECTRON TUBES	TUBES PER 1000 DIGITS	WORDS	TRACK GROUPS	ELECTRON TUBES	TUBES PER 1000 DIGITS
61,440	8	4.3	10	71	4,096	4	500	8.1	2,048	2	630	10	1,024	1	910	15
122,880	8	4.3	18	131	8,192	8	690	5.6	4,096	4	820	6.7	2,048	2	1090	8.9
245,760	8	4.3	33	251	16,384	16	1080	4.4	8,192	8	1200	4.9	4,096	4	1460	5.9
122,880	16	8.5	10	72	8,192	4	510	4.2	4,096	2	640	5.2	2,048	1	920	7.5
*245,760	*16	*8.5	*18	*132	16,384	8	700	2.8	*8,192	*4	*830	*3.4	4,096	2	1110	4.5
491,520	16	8.5	33	252	32,768	16	1090	2.2	16,384	8	1210	2.5	8,192	4	1480	3.0
245,760	32	17	10	73	16,384	4	520	2.1	8,192	2	650	2.6	4,096	1	930	3.8
491,520	32	17	18	133	32,768	8	710	1.4	16,384	4	840	1.7	8,192	2	1120	2.3
983,040	32	17	33	253	65,536	16	1100	1.1	32,768	8	1230	1.3	16,384	4	1490	1.5
491,520	64	34	10	74	32,768	4	530	1.1	16,384	2	660	1.3	8,192	1	940	1.9
983,040	64	34	18	134	65,536	8	720	0.73	32,768	4	850	0.87	16,384	2	1130	1.2
1,966,080	64	34	33	254	131,072	16	1110	0.56	65,536	8	1240	0.63	32,768	4	1500	0.76

The timing pulses in the storage section need not be synchronized with the clock or timing pulses in other portions of the IPS, since all digital and control information transmitted to the storage section is received and temporarily held in toggle-circuits. Information transmitted from the storage section is received on a toggle-circuit or relay register at the destination, which provides similar buffer storage. The property of asynchronism obviates the need for precise control of the angular velocity of the drum.

Characteristics of Typical Systems.—The principal characteristics of 36 typical drum storage systems are listed in Table I. These examples are all

similar to the one shown in the block diagram of Figure 1, with variations in access time, storage capacity, and word size. All designs are based on a common set of physical parameters: 128 digital cells scanned by each head per millisecond; 80 digital cells per inch of track; and 8 tracks per axial inch of drum.

Each of the 12 horizontal lines of the table corresponds to a drum of given diameter and length. Four values of diameter and three values of length are represented. The four diameters correspond to drum rotation periods or maximum access times of 8, 16, 32, and 64 milliseconds. The three lengths are for drums having 60, 120, and 240 storage tracks (in addition to angular index and timing tracks). Each line of the table contains characteristics of three systems corresponding to word sizes of 15, 30, and 60 binary digits. Characteristics of the particular example described in connection with Figure 1 are identified in the table by asterisks.

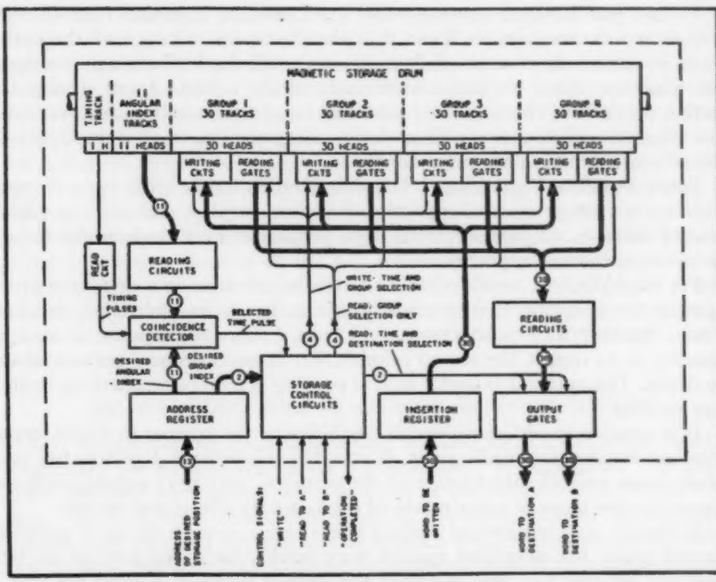

FIG. 1. Functional Block Diagram of Magnetic Drum Storage System.

The table contains information as to the number of magnetic heads and the number of electron tubes required for each example. It will be noted that the number of tubes is essentially constant for a given word size and drum length. Under these conditions the access time and storage capacity are both directly proportional to drum diameter.

Circuit cost per unit storage capacity may be expressed in terms of the number of tubes per thousand binary digits. This quantity is seen to be a decreasing function of access time and storage capacity, each considered independently, and a moderately increasing function of word size.

An idea of the space occupied by a given storage system may be gained from the tabulated data. The size of each drum is given in the table. The size of the cabinets needed to contain the electronic circuits may be estimated by allowing about one cubic foot for every 30 tubes.

Loading of Drum.—The contents of storage undergo numerous changes during the course of operation of an IPS. The entering of initial contents and the introduction of new data at occasional intervals is a function of the input section of the IPS. The choice of the input medium is governed largely by the application. Input data may be on magnetic tape or wire, punched cards, punched paper tape, or perhaps even introduced manually from a keyboard. The present storage system is capable of accepting successive items at rates up to about 500 *words* per second.

An input system using punched paper tape as the medium has been developed for use with a storage system similar to the example of Figure 1. The tape is scanned by a photoelectric reading device at a nominal speed of 75 feet per minute, corresponding to a storage insertion rate of 1800 30-digit words per minute. Even if it should be desired to load the entire drum, it would take only about five minutes to fill the 8192 storage positions. The magnetic drum rotates continuously at its normal speed during the loading operation. The tape feed need not be synchronized with drum rotation. Simple means are provided for loading sequences of data into any desired storage positions, in any order.

Some Possible Variations.—The described function table type of magnetic drum storage system embodies only the simplest and most straightforward features. Departures from these properties may be desirable to suit the needs of certain applications.

For example, it is possible to shorten the access time in a system of given capacity by assigning two or more magnetic heads to each track in place of one. Another way to shorten access time, but at the expense of storage capacity, is to repeat the stored information in several equal sectors about the drum. This method is useful only if reading is a more frequent operation than writing.

It is possible to add considerable flexibility to the manner in which stored items are located for reading out. If suitable coincidence detectors are provided, items written into storage in the standard way may subsequently be located on the basis of certain sets of digits *within* the stored words.

Although communication within the storage section is on a parallel-channel basis, the described system may readily be made part of an IPS operating serially, i.e., one in which the several digits of a word are transmitted sequentially over a single channel. This requires that the Address Register and the Insertion Register be endowed with the property of shifting. The incoming word then arrives digit by digit at one end of the receiving register. The register shifts its contents by one place upon arrival of each digit, until the complete word is assembled. Shifting registers must also be provided for transmitting words out of the storage section in serial fashion.

Status.—The developmental status of the magnetic drum storage technique at the time of this writing (May 1949) may be summarized as follows. A complete pilot model of the described function table type of system is undergoing final tests. Although this model is scaled down in capacity and word size, the standardized values of the physical parameters are used, and

every basic system function is included. Tests of every operating function under every expected condition have been performed with a repeated reliability which confirms the adequacy of the selected design standards. Reliability of the basic circuits and of the magnetic and mechanical components has been further established in an extensive laboratory program of component research and in the development of other types of magnetic drum storage systems during the past two years.

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DISCUSSIONS

Notes on Modern Numerical Analysis—I

EDITORIAL NOTE: There is a general feeling that, once the problems of construction and maintenance of automatic digital computing machines are solved, the remaining problems will be relatively simple. This may be the case if attention is confined to standard classical problems; however, if an attempt is made to use these machines fully, one is likely to encounter formidable mathematical difficulties. It is expected that these difficulties will be discussed in the current mathematical journals; but there are also smaller, more technical problems which may cause trouble. It is believed that a discussion of these smaller problems will prove beneficial in avoiding a great many difficulties which are expected to arise when the machines are in actual operation; and we should like to urge interested persons to submit technical notes of this nature for future publication in the Automatic Computing Machinery Section of *MTAC*. These notes could be by-products of or preliminaries to more constructive investigations. It would be a great advantage, for expository purposes, if the authors, even at the expense of a choice of an extravagant example, could exhibit the troubles under discussion on a manual scale.

Solution of Differential Equations by Recurrence Relations

1.1. In general the most satisfactory method for the numerical solution of ordinary differential equations is one of the "extrapolation" methods.[1] These methods have proven very efficient in the hands of a practiced computer. There is little doubt that some of the experience he uses could be codified and adapted for use on automatic digital computing machines. Nevertheless, the use of some direct recursive process is very attractive and worth investigation.

Let us consider the solution by such methods of the equation

$$y'' = -y, \tag{1}$$

with the boundary conditions $y(0) = 0, y'(0) = 1$, by use of the well-known formula[2]

$$h^2y'' = (\delta^2 - \tfrac{1}{12}\delta^4 + \tfrac{1}{90}\delta^6 - \cdots)y. \tag{2}$$

1.2. First let $h = 1$, using only the first term on the right-hand side of (2). If the condition $y'(0) = 1$ is replaced by $y(1) = 1$ the following recurrence relation is obtained

$$y(n+2) = y(n+1) - y(n) \tag{3}$$

with the boundary conditions $y(0) = 0$, $y(1) = 1$. For $n = 0, 1, 2, 3, \cdots$,

$$y(n) = 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, \cdots. \tag{4}$$

Two points are now worth noting. One way of introducing the circular functions analytically is to define $\sin x$ as the solution of (1); in this treatment π is defined as the least positive root of $\sin x = 0$. Observe that 3 has now been obtained as an approximation to π. Secondly, it is seen that the solution (4) is periodic.

1.3. By taking a smaller value of h, a possible improvement can be expected. If we take $h = 0.1$, the recurrence relation is now

$$y(n + 2) = 1.99y(n + 1) - y(n) \tag{5}$$

with $y(0) = 0$ and $y(1) = \sin 0.1 = 0.09983$. The solution obtained when five decimal places are used is given in column (5) of Table I and may be compared with the corresponding values of $\sin x$, to ten places, given in column (1). It will be seen that for $0 \leq x \leq 16$ the error is always negative and steadily increases in absolute value, being $-35 \cdot 10^{-5}$, when $x = 1.6$.

This solution is apparently not periodic, and we may inquire as to the existence of values of h (other than $h = 1$) for which the solution is periodic, i.e., for which the sequence of its values is periodic. It can be shown that the only values of h are $h = 2 \sin \pi/n$ for $n = 2, 3, \cdots$. When $h = 2 \sin \pi/n$, the period is n and the corresponding approximation to π is $n \sin \pi/n$ which is roughly $\pi[1 - (6n^2)^{-1}]$.

1.4. Let us next consider the possibility of improving the solution by using two terms on the right-hand side of (2). As previous experience has taught us the benefit of taking two further differences into account, it would seem plausible to expect a marked improvement. In fact, however, if we take $h = 0.1$ and work to ten places, using the natural boundary conditions, the solution of the corresponding difference equation $(\delta^2 - \frac{1}{12}\delta^4)y = -0.01y$, i.e.,

$$y(n + 4) = 16y(n + 3) - 29.88y(n + 2) + 16y(n + 1) - y(n) \tag{6}$$

rapidly diverges to $+\infty$, as is seen in column (6). The same behavior occurs if 9, 8, 7, or 6 places are used, but if 5 places are used, it will be found [see column (7)] that the solution rapidly tends to $-\infty$.

1.5. If use is made of the equation obtained by neglecting all but the first two terms on the right-hand side of (2) and substituting in (1), we find

$$(\delta^2 - \tfrac{1}{12}\delta^4)y = -h^2y. \tag{8}$$

If the term δ^4y on its left is replaced by its approximate value

$$-h^2\delta^2y \sim \delta^2(\delta^2y), \tag{9}$$

a three-term relation is obtained

$$(12 + h^2)\delta^2y = -12h^2y. \tag{10}$$

Using $h = 0.1$, the following equation replaces (6)

$$y(n + 2) = 1.99000\,83333y(n + 1) - y(n). \tag{11}$$

TABLE I

x	(1)	(5)	(6)	(7)	(11)
0	0.00000 00000	0.00000	0.00000 00000	0.00000	0.00000 00000
0.1	0.09983 34166	0.09983	0.09983 34166	0.09983	0.09983 34166
0.2	0.19866 93308	0.19866	0.19866 93308	0.19867	0.19866 93310
0.3	0.29552 02067	0.29550	0.29552 02067	0.29552	0.29552 02077
0.4	0.38941 83423	0.38939	0.38941 83685	0.38934	0.38941 83450
0.5	0.47942 55386	0.47939	0.47942 59960	0.47819	0.47942 55440
0.6	0.56464 24734	0.56460	0.56464 90616	0.54721	0.56464 24828
0.7	0.64421 76872	0.64416	0.64430 99144	0.40096	0.64421 77021
0.8	0.71735 60909	0.71728	0.71864 22373	−2.67357	0.71735 61128
0.9	0.78332 69096	0.78323	0.80125 45441		0.78332 69403
1.0	0.84147 09848	0.84135	1.09135 22239		0.84147 10261
1.1	0.89120 73601	0.89106	4.37411 56871		0.89120 74139
1.2	0.93203 90860	0.93186			0.93203 91543
1.3	0.96355 81854	0.96334			0.96355 82701
1.4	0.98544 97300	0.98519			0.98544 98328
1.5	0.99749 49866	0.99719			0.99749 51092
1.6	0.99957 36030	0.99922			0.99957 37469

The solution of this equation, using the natural boundary conditions, is shown in column (11) of the table. The error is positive and steadily increases to the value 1439×10^{-10} at $x = 1.6$. The device used here is well known and is the basis of the NUMEROV-MILNE method for the solution of second order differential equations.

1.6. The reason for the surprising results mentioned in **1.4** is clear; they are essentially due to the large coefficient 16 of $y(n + 3)$ in (6). Assuming that $y(0)$, $y(1)$, $y(2)$, and $y(3)$ are correct, apart from rounding off errors, we may expect an error in $y(4)$ due either to the rounding off of the given values or to the truncation error caused by neglect of the terms involving the sixth and higher differences in (2) or to both these causes. These errors can easily be estimated. The first error cannot exceed $(16 + 29.88 + 16 + 1)\cdot\frac{1}{2}\cdot 10^{-r}$ which is about $3\cdot 10^{-r+1}$ when one is working to r decimal places. The second error may be estimated as

$$- 12\cdot\tfrac{1}{90}\delta^6 y - 1.\dot{3}\times 10^{-7} y$$

which is about $2.\dot{6}\times 10^{-8}$, when $x = 0.2$. When 10 decimal places are used, the truncation error is the dominant one. Examination of the rounding off errors caused by carrying out $\sin x$ (for $x = 0.1$, 0.2, and 0.3) to 9, 8, 7, or 6 places shows that the resultant error in $y(4)$ is positive, but if 5 places are used, it is negative and greater in magnitude than the truncation error.

It is this initial error which determines the ultimate behavior of $y(n)$. The error in $y(5)$ is roughly 16 times that occurring in $y(4)$, as the errors in $y(3)$, $y(2)$, $y(1)$ are negligible compared with that in $y(4)$; the error in $y(6)$ is roughly 16 times that occurring in $y(5)$, for similar reasons; and so on. This exponential increase in the error serves as the explanation of the observed divergences. It is important to note that the trouble has been caused not by an accumulation of rounding off errors but rather by a single error (caused either by rounding off or truncation) and an unfortunate choice of formula.

Let us examine this more precisely. The solution of the difference equa-

tion (6) is of the form

$$y(n) = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + A_4\alpha_4^n$$

where α_1, α_2, α_3, α_4 are the roots of the equation

$$x^4 - 16x^3 + 29.88x^2 - 16x + 1 = 0.$$

This equation has two real roots α_1, α_2 and two complex roots α_3, α_4. Since it is a reciprocal equation with real coefficients, the following relations must hold

$$\alpha_1\alpha_2 = 1, \quad \alpha_3\alpha_4 = 1, \quad \alpha_3 = \bar{\alpha}_4.$$

The solution is therefore of the form

$$y(n) = A\alpha^n + B\alpha^{-n} + C\cos n\theta + D\sin n\theta,$$

where $\alpha = \alpha_1$, and $\theta = \arg \alpha_3$. The following approximate values are found

$$\alpha = 13.94, \qquad \theta = 0.1000.$$

It is clear that, if $A \neq 0$, then the term $A\alpha^n$ will dominate the others in the solution.

It will be found that the ratio of successive errors in column (6) approaches 13.94 very rapidly.

1.7. The solution of the difference equation (10) is

$$y(n) = A\cos n\theta + B\sin n\theta$$

where, for general h, θ is defined by

$$\cos\theta = 1 - [6h^2/(12 + h^2)].$$

We are interested in the pure sine solution. A small error will introduce a component $A\cos n\theta$ which will remain small. Significant results are to be expected in this case although there will, in general, be an error caused by the accumulation of the rounding off error or by the truncation of the formula (2).

Some idea of the magnitude of the first kind of error can be obtained by consideration of the difference equation arising from (10) with $h = 0$,

$$y(n + 2) = 2y(n + 1) - y(n).$$

There is no rounding off here, since the coefficients are integral. For this equation, we have

$$y(n) = (-n + 1)y(0) + ny(1) + (n - 1)y(2) + (n - 2)y(3) + \cdots + 2y(n - 1).$$

If we denote by $e(n)$ the error committed in rounding off the right-hand side of the last equation, then the total error in $y(n)$ due to rounding off is

$$(-n + 1)e(0) + ne(1) + (n - 1)e(2) + (n - 2)e(3) + \cdots + 2e(n - 1) + e(n).$$

Assuming that the $e(r)$'s, where $r = 0, 1, 2, 3, \cdots n$, are independent and have a rectangular distribution, theu, for large values of n, the distribution

of the total error in $y(n)$ is approximately normal with zero mean and variance

$$\sigma^2 = \tfrac{1}{12}\{(n-1)^2 + n^2 + (n-1)^2 + (n-2)^2 + (n-3)^2 + \cdots + 2^2 + 1^2\} \simeq n^3/36.$$

The probable total error is thus, in units of the last decimal,

$$0.6745\sigma \simeq 0.6745n^{3/2}/6, \text{ i.e., about } 0.1n^{3/2}.$$

The maximum error of this kind cannot exceed

$$\tfrac{1}{2}\{(n-1) + n + (n-1) + (n-2) + (n-3) + \cdots + 2 + 1\} \simeq \tfrac{1}{4}n^2$$

units of the last decimal.

Some idea of the magnitude of the truncation error is obtained by noticing that, on the assumption that no rounding off errors are committed, the solution obtained is

$$y(n) = \sin n\theta = \sin nh[1 + (h^4/480) + O(h^6)]$$

instead of $y(n) = \sin nh$.

The main source of error in column (11) is due to truncation while that in column (5) is due to rounding off.

1.8. The solution of a differential equation of the form

$$y'' = -k^2y,$$

where k is a constant, can be discussed in exactly the same way as we have dealt with the case where $k = 1$. Similar considerations will apply to

$$y'' + I(x)y = 0,$$

in the oscillatory regions, i.e., where $I(x)$ is positive. In the exponential regions, where $I(x)$ is negative, there will be solutions which diverge and solutions which converge. Contamination of a divergent solution with a small component of a convergent will, in general, cause no serious trouble, but the reverse effect must be avoided, e.g., by working backwards so that the recurrence relations are used in the form

$$y(n) = a_1y(n+1) + a_2y(n+2) + a_3y(n+3) + \cdots + a_ry(n+r).$$

The behavior of the solutions in the transition case near (simple) zeros of $I(x)$ will, in general, be on the pattern of those of

$$y'' = xy$$

which are the Airy Integrals, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.[3]

1.9. Recently L. Fox and E. T. Goodwin [4] suggested the following type of method for the practical solution of ordinary differential equations with one-point boundary conditions: Use a recurrence relation of the form of (5) or (11) to obtain an approximation to the solution. Difference this solution and use the differences in the untruncated formula to correct the solution at each stage of the recursion. Difference again and correct again. Repeat until satisfactory solutions are obtained. The process appears likely to be of considerable use, but when using it, as was done in this instance, care must be taken in the choice of the recurrence relation used.

1.10. Similar phenomena for partial differential equations have been investigated by L. COLLATZ,[5] who indicates some conditions under which recurrence relations can be useful in the solution of such problems.

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[1] See, e.g., L. J. COMRIE, *Chamber's Six-figure Mathematical Tables*, London, v. 2, 1949, p. 545–549 and *Interpolation and Allied Tables*, H. M. Stationery Office, London, 1947, p. 942–943.

[2] See, e.g., W. E. MILNE, *Numerical Calculus*, Princeton Univ. Press, 1949, p. 192.

[3] See, e.g., J. C. P. Miller, *The Airy Integral*, Brit. Assn. Math. Tables, Part vol. B, Cambridge Univ. Press, 1946.

[4] L. Fox & E. T. GOODWIN, "Some new methods for the numerical integration of ordinary differential equations," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 373–388.

[5] L. COLLATZ, Über das Differenzenverfahren bei Anfangswertproblemen partieller Differentialgleichungen," *ZAMM*, v. 16, 1936, p. 239–247.

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4. ANDREW D. BOOTH, "A magnetic digital storage system," *Electronic Engineering*, v. 21, July 1949, p. 234–238, 19 figs. 26 × 19 cm.

5. E. M. DEELEY and D. M. MACKAY, "Multiplication and division by electronic-analogue methods," *Nature*, v. 163, Apr. 23, 1949, p. 650. 18 × 27.3 cm.

6. HERMAN H. GOLDSTINE and JOHN VON NEUMANN, *Planning and Coding of Problems for an Electronic Computing Instrument*, Institute for Advanced Study, Princeton, N. J., part II, v. 2, 1948, 68 pages, 7 figs. 21.6 × 27.9 cm.

The second volume on the planning and coding of problems for an electronic computer illustrates the preparation of several problems for the proposed Institute for Advanced Study electronic computing machine. The many steps, from the initial statement of the problem and its mathematical foundation through the final coding of the problem in machine language, are given in complete detail.

The first volume of this report [see *MTAC*, v. 3, p. 54] presented the fundamental information and explanations necessary to the understanding of the publication now under review, i.e., the instruction code of the IAS computer, the preparation of flow diagrams, and several examples of the coding of some basic arithmetic operations. The problems encountered in the second volume are numerical integration, interpolation schemes, sorting, and collating.

Automatically-sequenced computing machines are most efficiently utilized when the problem to be computed can be reduced to an iterative process. The report gives rigorous treatment to the reduction of problems to such form, which lends itself to easy translation into machine language. There

are two grave difficulties which must be considered when working with computers having a fixed decimal point, which maintain the same range for resultant values as for the operands. They are (1) round-off errors, and (2) the possibility of overflow, i.e., of exceeding the magnitude of the numbers accepted by the machine. The first of these is competently considered by the authors—with an especially efficient treatment of the method derived for applying Lagrange's interpolation formula. The second difficulty is presumed to have been avoided by the application of scale factors to the problem data prior to their insertion into the computer. Although this can be accomplished easily for the problems considered here, there are many others which are too complex for such treatment and require a more complicated routine, in order to accommodate a floating decimal point.

The mathematical discussions presented as a preliminary, though most important, phase of preparing problems for an automatic computer are generally applicable to all of the high-speed machines presently contemplated or under construction. For this reason the report constitutes a valuable instruction manual for all persons interested in programming problems for a high-speed computer.

The subsequent steps of preparing flow diagrams and coded routines are especially designed to meet the specifications of the IAS machine. The authors, in their attempt to meet the requirements of an instruction manual (i.e., starting with the simple specific case and then progressing to the more difficult general case) have introduced certain inefficiencies into the method of programming and coding employed. In practice, the MDL has found it to be more efficient, both in terms of memory space and time of execution, to reverse the procedure and introduce the parameters for the specific problem into the general routine, rather than to program the modifications necessary to make the simpler routine more generally applicable.

Also, the attempt to standardize certain procedures and notations, while helpful to the novice, precludes the introduction of certain economies in coding, e.g., the notation $x_s = 2^{-19}x + 2^{-39}x$, used for the storage of all parameters.

The increased efficiency made possible by deviating from the authors' procedure, as indicated in the previous two points, is well illustrated in Dr. SAMUEL LUBKIN's review of volume 3 of the report [*MTAC*, v. 3, p. 541–542]. He points out that the number of words required for the problem contained in that volume can be reduced from $58 + 4I$ to $36 + 2I$.

The first two problems discussed in this volume treat the integration of a tabulated one-variable function. One method makes use of SIMPSON's formula for the limited case of $\int_0^1 f(z)dz$; the other method involves a more general formula and integrates

$$\int_{(\lambda-\frac{1}{2})/N}^{(k-\frac{1}{2})/N} f(z)dz,$$

where k and λ may be any positive integers. It appears likely that, in a great many cases, scaling could be introduced in the function values to permit summing of terms having the same coefficients without exceeding capacity, and the common factor could be applied only once for each case, with resulting gain in time. Since this problem, as stated, does not assume

such scaling, it is necessary to multiply each term of the summation individually. The authors accomplish this by taking each term in sequence as derived in the formula. By grouping those with like coefficients, however, many fewer modifications and comparisons need be programmed. Another possibility is to "spell out" the few alternatives, instead of modifying the common sequence for each successive term.

The next group of problems deals with the application of Lagrange's interpolation formula. They are:

(1) Given variables $x_1, \cdots, x_M (x_1 < x_2 < \cdots < x_M)$ and the function values $p_1, \cdots, p_M$, to interpolate this function for the value x of the variable.

(2) Given the tabulated function $p_1, \cdots, p_N$, to interpolate for the value of x, using the M points nearest x, when the values of $x_1, \cdots, x_N$ are equidistant.

(3) Same as (2) above, except that the independent variables are unrestricted.

(In problems (1) and (3) it is assumed that the variable function values are stored in two separate sequences.)

(4) Same as (3), except that the x's and p's are intermeshed in one sequence of $2N$ cells.

Although the mathematical formulation as derived by the authors is, in the opinion of this reviewer, exceptionally well-fitted for machine computation, the actual coding reflects the same general inefficiencies stated earlier. The problems (2) through (4) are obtained by adding the required modifications to the limited first case, a circumlocutory approach which is needlessly time- and space-consuming.

The last two problems are:

(1) Meshing. (Given two monotonic sequences of n and m complexes: to merge them into one monotonic sequence.)

(2) Sorting. (Given a random sequence of complexes: to arrange them in monotonic order.)

A complex $X = [x, u_1, \cdots, u_p]$ consists of $(p + 1)$ words. A monotonic sequence of complexes is arranged in the order of the principal number x. Sorting is accomplished by considering the initial data as N sequences of one complex each and obtaining $N/2$ monotonic sequences of two complexes each. This process is continued until all of the information has been arranged in one monotonic sequence. It is evident that the first problem can be treated as a special case of the second. However the general routine again is wasteful of machine time and space.

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NBSMDL

7. Harvard University, Computation Laboratory, *Annals*, v. 14, *Description of a Relay Calculator*, Cambridge, Mass., Harvard University Press, 1949, xvi, 366 pages, 36 plates, \$8.00. 20 × 26.5 cm.

When a group of scientists has completed a project, it is faced with the job of reporting its results. Too often the effect of the report on both writer and reader is that of turkey hash the week after Thanksgiving. The Harvard Computation Laboratory, however, has made a sincere effort to write useful

and well-planned descriptions of its computers. Following its *Manual of Operation* for the Mark I Computer, the Laboratory has published a description of the Mark II Relay Computer. The description is, of course, a technical report, unadorned with facile generalities. It does, however, list the names of the men and women who have contributed to the technical work. In the opinion of the reviewer, this acknowledgment is a desirable break from the attitude of many organizations that insist on Siberian isolation for scientific employees.

The reader who is primarily interested in keeping up with the expanding universe of automatic computers will want to study Chapter I entitled *The Organization of the Calculator*, with its 21 photographic plates showing the components of the computer and the general view of the machine. Those who intend to use the calculator will also want to learn about the *Operation of the Calculator* (Chapter X), *Problem Preparation* (Chapter XI), and the handling of the elementary functions (x^{-1}, $1/\sqrt{x}$, $\log_{10} x$, 10^x, $\cos x$, and $\arctan x$) which is discussed in Chapter VII.

Most of the remaining chapters are written for the engineer or mathematician who will operate this machine or who is actively working on the design of automatic computers. These chapters treat basic circuits, registers (memory), addition units, multiplication units, sequencing and control, interpolators, and input and output devices.

It may be interesting to compare the Harvard Relay Calculator, Mark II, with the five relay computers of various sizes already built by the Bell Telephone Laboratories. Of the latter, the so-called "Stibitz Computers" located at Langley Field and Aberdeen Proving Ground, are very nearly the same size as the Mark II, each of the three machines having about 12,000 relays. Each machine is divided into two calculators that can be used separately or can be combined to work on one problem. Each uses a "floating decimal" system, writing every number as a set of significant digits multiplied by a power of 10 so that the largest number of significant digits may be retained at each step.

Some of the outstanding differences are as follows:

(1) The BTL machines are self-cycled, i.e., the completion of each minor step signals the next step to start, whereas the Mark II is time-cycled, and each minor step is allotted a fixed time interval.

(2) The BTL machines represent decimal digits in a "bi-quinary" notation, with 7 relays. In this notation one relay of a pair and one of a set of five are always closed when a digit is stored. If more or less than this complement operate, a check circuit stops the machine. Mark II uses a straight binary designation with 4 relays for each decimal digit.

(3) The BTL computers carry 7 decimal places, and Mark II carries 10.

(4) The BTL computers use practically no specially-made equipment, being built almost entirely of standard telephone and teletype units, whereas the designers of Mark II have felt free to design and build special devices.

In general, it may be said that the speeds of operation and the capabilities of the BTL and Mark II computers are similar, but the engineering is distinctive. It will be interesting to see whether the operating experience will be essentially different and, in particular, whether the expenditure of 3

extra relays per digit to obtain self-checking in the BTL computers is worthwhile.

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8. H. R. Hegbar, "Electronic analog computer," *Electronics*, v. 22, Mar. 1949, p. 168, 170, 172, 174, illustr. and diag. 20.3 × 29.8 cm.

9. Frank A. Metz, Jr., and Walther M. A. Andersen, "Improved ultrasonic delay lines," *Electronics*, v. 22, July 1949, p. 96–100, bibl. 20.3 ×29.8 cm.

Forged magnesium-alloy delay lines developed as memory devices have bandwidths as great as 4 mc. at a carrier frequency of 10 mc. The attenuation is the least so far available in practical lines. Special clamping of S-cut ADP crystal transducers is described.

10. C. H. Page, "Digital computer switching circuits," *Electronics*, v. 21, Sept. 1948, p. 110–118, 4 figs. 20.3 × 29.8 cm.

11. Louis N. Ridenour, "Mechanical brains," *Fortune*, v. 39, May 1949, p. 109–118, illustrs. 33 × 25.8 cm.

12. T. K. Sharpless, "High-speed *n*-scale counter," *Electronics*, v. 21, Mar. 1948, p. 122–125, illustrs., diags., bibl. 20.3 × 29.8 cm.

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News

Eckert-Mauchly Computer Corporation.—The BINAC (Binary Automatic Computer) made its debut in the latter part of August at the Eckert-Mauchly Corporation, in Philadelphia, Pa. This all-electronic machine was built for Northrop Aircraft, Inc., of Hawthorne, California, and embodies a considerable number of improvements over the first such machine, the Army's 30-ton ENIAC, for whose construction John W. Mauchly and J. Presper Eckert, Jr., were largely responsible.

Although the BINAC actually consists of twin machines, working in unison and checking each other's performance, it is small enough to fit into an ordinary-sized room, as each machine stands five feet high and occupies less than ten square feet of floor space. The most important of its many novel features is its mercury delay line "memory," whose invention contributed enormously to the reduction in the number of electronic tubes originally found necessary in the construction of the ENIAC, which is burdened with 18,000 such tubes. Each BINAC twin possesses only 700 tubes, although it is equipped with 512 memory registers (or "cells"), each capable of storing either a signed number consisting of thirty binary digits or a pair of "instructions."

The machine is constructed to follow a set of sixteen coded instructions for the execution of elementary mathematical operations. A staff of programmers and coders arrange these instructions in the proper sequence for carrying out a required computation. The resulting routine, together with the original data, is then transferred onto magnetic tape by means of a specially constructed typewriter which transforms the original material into binary-coded decimal form while yielding simultaneously a typed paper copy of the transcribed material, in the original form. By means of a manually operated switch, the information on the magnetic tape is inserted into the memory of the machine.

The very high speed, at which instructions and data are delivered from the memory to

the other units within the machine and at which arithmetical operations are executed, is due to the enormous rapidity of the BINAC's "heart-beat"—its oscillator emits four million pulses every second.

During the demonstration, the BINAC gave repeat performances for two types of problems. First, the solution of a Poisson equation was computed by means of a modified Liebmann method, over a square field with 16^2 mesh points. Each transit over the 196 interior points consumed about four seconds of the machine's time. The largest difference between two successive results obtained for each point was recorded on the oscilloscope, and the computation stopped when the preassigned tolerance exceeded this difference. The entire solution, correct to about 8 decimal places, took less than ten minutes per problem.

The second part of the demonstration was devoted to obtaining the reciprocal, the square root, the reciprocal square root, and the cube root of four-decimal-digit numbers suggested by the audience. In addition to converting the original numbers into their binary equivalents to enable the Arithmetic Unit to compute the required functions, the machine also sorted the numbers, by collation, in ascending order of magnitude. The audience was invited to compare the answers with the entries in BARLOW's Tables.

Harvard University Computation Laboratory.—On September 13–16, 1949, a second Symposium on Large-Scale Digital Calculating Machinery was held at the Computation Laboratory of Harvard University under the joint sponsorship of the Navy Department's Bureau of Ordnance and Harvard University. In conjunction with this Symposium there was also a meeting of the Association for Computing Machinery. The eight sessions of the Symposium were supplemented by a continuous demonstration of the Mark III Calculator. The program was as follows:

Tuesday Morning, September 13, 1949, *Opening Addresses*, HOWARD H. AIKEN, Director of the Computation Laboratory, presiding, EDWARD REYNOLDS, Administrative Vice-President of Harvard University, and Rear Admiral F. I. ENTWISTLE, USN, Director of Research, Bureau of Ordnance.

Tuesday Afternoon, *Recent Developments in Computing Machinery*, MINA REES, Office of Naval Research, presiding:

"The Mark III calculator," BENJAMIN L. MOORE, Harvard University

"The Bell Computer, Model VI," ERNEST G. ANDREWS, Bell Telephone Laboratories

"Tests on a dynamic regenerative electrostatic memory," J. PRESPER ECKERT, JR., Eckert-Mauchly Computer Corporation

"The digital computation program at Massachusetts Institute of Technology," JAY W. FORRESTER, Massachusetts Institute of Technology

"The Raytheon electronic digital computer," RICHARD M. BLOCH, Raytheon Manufacturing Company

"A General Electric engineering digital computer," BURTON R. LESTER, General Electric Company.

Wednesday Morning, September 14, 1949, *Recent Developments in Computing Machinery*, E. LEON CHAFFEE, Harvard University, presiding:

"Design features of the NBS Interim and Zephyr Computers," S. N. ALEXANDER and H. D. HUSKEY, National Bureau of Standards

"Static magnetic delay lines," WAY DONG WOO, Harvard University

"Coordinate tubes for use with electrostatic storage tubes," R. S. JULIAN and A. L. SAMUEL, University of Illinois

"Basic aspects of special computational problems," HOWARD T. ENGSTROM, Engineering Research Associates

"Electrochemical computing elements," JOHN R. BOWMAN, Mellon Institute

"EDVAC transformation rules," GEORGE W. PATTERSON, University of Pennsylvania.

Wednesday Afternoon, *Recent Developments in Computing Machinery*, RAYMOND C. ARCHIBALD, Brown University, presiding:

"Notes on the solution of linear systems involving inequalities," GEORGE W. BROWN, Rand Corporation

"Mathematical methods in large-scale computing units," D. H. LEHMER, University of California

"Empirical study of effects of rounding errors," C. CLINTON BRAMBLE, U. S. Naval Proving Ground, Dahlgren, Va.

"Numerical methods associated with Laplace's equation," W. E. MILNE, Institute for Numerical Analysis, UCLA and Oregon State College

"An iteration method for the solution of the characteristic value problem of linear differential and integral operators," CORNELIUS LANCZOS, Institute for Numerical Analysis, UCLA

"The Monte Carlo Method," S. M. ULAM, Los Alamos Scientific Laboratory.

Thursday Morning, September 15, 1949, *Computational Problems in Physics*, KARL K. DARROW, Bell Telephone Laboratories, presiding:

"The place of automatic computing machinery in theoretical physics," WENDELL FURRY, Harvard University

"Double refraction of flow and the dimensions of large asymmetrical molecules," HAROLD A. SCHERAGA and JOHN T. EDSALL, Cornell University and Harvard Medical School

"L-shell internal conversion," MORRIS E. ROSE, Clinton Laboratories, Oak Ridge

"The use of calculating machines in the theory of primary cosmic radiation," MANUEL S. VALLARTA, University of Mexico

"Computational problems in nuclear physics," HERMAN FESHBACH, Massachusetts Institute of Technology.

Thursday Afternoon, *Aeronautics and Applied Mechanics*, HARALD M. WESTERGAARD, Harvard University, presiding:

"Computing machines in aeronautical research," R. D. O'NEAL, University of Michigan

"Problem of aircraft dynamics," EVERETT T. WELMERS, Bell Aircraft Corporation

"Statistical methods for certain non-linear dynamical systems," GEORGE R. STIBITZ, Burlington, Vermont

"Combustion aerodynamics," HOWARD W. EMMONS, Harvard University

"Application of computing machinery in research of the oil industry," MORRIS MUSKAT, Gulf Research and Development Company

"The 603–405 Computer," WILLIAM W. WOODBURY, Northrop Aircraft, Inc.

Friday Morning, September 16, 1949, *The Economic and Social Sciences*, EDWIN B. WILSON, Office of Naval Research, presiding:

"Application of computing machinery to the solution of problems of the social sciences," C. FREDERICK MOSTELLER, Harvard University

"Dynamic analysis of economic equilibrium," WASSILY W. LEONTIEF, Harvard University

"Some computational problems in psychology," LEDYARD TUCKER, Educational Testing Service, Princeton

"Computational aspects of certain econometric problems," HERMANN CHERNOFF, University of Chicago

"Physiology and computing devices," WILLIAM J. CROZIER, Harvard University

"The science of prosperity," FREDERICK V. WAUGH, Council of Economic Advisers.

Friday Afternoon, *Discussion and Conclusions*, WILLARD E. BLEICK, U. S. Naval Academy Post Graduate School, presiding:

"Computer built by the Centre Blaise Pascal," LOUIS COUFFIGNAL, Institut Blaise Pascal (read by LEON BRILLOUIN)

"The future of computing machinery," L. N. RIDENOUR, University of Illinois.

This meeting, attended by well over 500 persons, is noteworthy in that its program dealt with not only those physical sciences which are already recognized as closely akin to the large-scale computer development, but also with the increasing application of these machines to the social and economic sciences. The hosts are to be congratulated upon the cleverly-conceived meeting in which topics covering widely diverse fields were organized into a well-integrated program. Each lecture complemented the related lectures without duplication of subject matter. It was interesting to note that many of the demon-

strators of the Mark III were students from foreign lands now working at the Computation Laboratory.

On Tuesday evening, September 13, 1949, a banquet was held with EDWARD A. WEEKS, JR., Editor of *The Atlantic Monthly*, as toastmaster, who introduced the speech of WILLIAM S. ELLIOTT, Elliott Brothers Research Laboratories (London) Limited, entitled "Present position of computing machine development in England."

The Cambridge University Mathematical Laboratory.—A four-day conference on automatic calculating machines held June 22 through June 25, 1949, inclusive, at the University Mathematical Laboratory in Cambridge, England, served to bring together some 150 scientists interested in the design and application of high-speed automatic calculating machines. Attention was naturally concentrated on developments in England and on the continent, although recent American developments were summarized in a paper expressly prepared for the conference by H. D. HUSKEY, National Bureau of Standards, and presented for him by J. M. BENNETT. References to American work were also made by D. R. HARTREE in a survey of the present position of work in the field and in a paper on relay machines by A. D. BOOTH, which was presented by Miss K. H. V. BRITTEN. There was a demonstration of the new Cambridge electronic calculating machine, the EDSAC, during which tables of squares and prime numbers were printed.

The full program of the conference was as follows:

Wednesday, June 22:	
Address of Welcome	M. V. WILKES, Director of the University Mathematical Laboratory
Survey of the present position of work on automatic digital computers	D. R. Hartree, Cavendish Laboratory
The EDSAC	M. V. Wilkes
Demonstration of the EDSAC	W. RENWICK, University Mathematical Laboratory
Thursday, June 23:	
The Automatic Relay Calculator	A. D. Booth, Birkbeck College. Paper presented by Miss K. H. V. Britten, British Rubber Producers' Research Association
Discussion on relay machines	
Cathode-ray tube storage	F. C. WILLIAMS, University of Manchester
Discussion on programming and coding	J. H. WILKINSON, N. P. L., presiding
French computing machine projects	L. COUFFIGNAL, Laboratoire de Calcul Mécanique, Institut Blaise Pascal, Paris
Friday, June 24:	
Checking process for large routines	A. TURING, University of Manchester
Some routines involving large integers	M. H. A. NEWMAN, University of Manchester
Discussion on permanent and semi-permanent storage facilities	E. N. MUTCH, University Mathematical Laboratory, presiding
Discussion on checking procedure and circuits	A. M. UTTLEY, T. R. E., and D. J. WHEELER, University Mathematical Laboratory, presiding
Saturday, June 25:	
Description of a machine built at Manchester	T. KILBURN, University of Manchester
Computing machines: plans, projects, and general ideas	General Discussion

In addition there were contributions from A. VAN WIJNGAARDEN, Mathematisch Centrum, Amsterdam, and G. KJELLBERG, Tekniska Högskolan, Stockholm.

Two electronic calculating machines are now in operation in England. One of these,

the EDSAC, is a serial binary machine using ultrasonic tanks for storage; it has a storage capacity of 512 words of 34 digits each plus a sign digit. Five-hole teleprinter tape is used for input and a modified teleprinter for output.

The other machine, located at Manchester, has grown out of the development of a "baby" machine built to test the practicability of the cathode-ray tube storage system developed by F. C. Williams and T. Kilburn. It now has about 1400 tubes. There is no printer, but the results are read from a cathode-ray tube connected to the store. In spite of these limitations, some genuine mathematical work has been done on Mersenne numbers. Addition takes 1.8 milliseconds and multiplication up to 36 milliseconds depending on the number of digits in the multiplier. An auxiliary store using a magnetic drum whose rotation speed is locked to the clock-pulses has been developed, and transfer of blocks of numbers or orders from it to the high-speed store is possible by manipulating a series of push buttons. This machine is in a constant state of change. Its main purpose is to provide experience for those working on computers.

The Automatic Relay Calculator which has been built under the direction of A. D. Booth is a relay machine working in the binary system. The store is a magnetic drum with capacity for 256 numbers. The machine has about 800 relays all of which are of the Siemens high-speed type and have an operating time of one or two milliseconds. The word length is 20 digits, plus a sign digit. Addition takes 20 milliseconds and multiplication takes 0.4 seconds. Teleprinter tape is used for input and a teleprinter for output. This machine is complete but is not yet in working order.

Of the other British computer projects, the most advanced is the construction of a pilot model for the Automatic Computing Engine at the National Physical Laboratory. This will be a binary machine with a word length of 32 digits and will use punched cards for input and output. The store will consist of a group of ultrasonic tanks with a total capacity of 256 words. The addition time will be 32 microseconds and the multiplication time 2 milliseconds.

Work is in progress at the Telecommunications Research Establishment of the Ministry of Supply on a parallel machine using cathode-ray tubes for the high-speed store and a magnetic drum for the auxiliary store. A special feature of this machine will be the use of three-state trigger circuits, two of the states being used to represent 0 and 1 and the third being neutral. Each trigger circuit will be put in the neutral state before each transfer operation, and thus definite action will be necessary to set it to represent either a 0 or a 1. In this way the machine can be made self-checking to a large extent.

A decimal relay machine using rotary switches for the registers is being constructed at the Royal Aircraft Establishment. Each word will consist of 8 decimal digits using a floating decimal point. The addition time will be about one second and the multiplication time 1½ to 2 seconds. The machine is intended primarily to facilitate the analysis of experimental results.

A new project of Dr. Booth's, to which the name APEXC (All-Purpose Electronic X-Ray Computer) has been given, was mentioned briefly. It will use a combination of relays and electronic tubes and will have a magnetic drum store with an electromechanical auxiliary store.

L. Couffignal gave some information about a parallel binary electronic machine which he plans to build at the Institut Blaise Pascal in Paris. Some preliminary design work has already been done, and the machine will probably have a word length of 50 digits with a floating binary point.

Dr. A. van Wijngaarden described a relay machine, rather similar to the Automatic Relay Calculator, being built at the Mathematisch Centrum at Amsterdam, Holland. This will be a parallel binary machine with a word length of 30 digits. The addition time will be 15 milliseconds and the multiplication time 0.4 seconds. It will have a magnetic drum store and teleprinter input and output equipment. Another relay machine is being built in Holland at the Technische Hogeskolan in Delft. This is primarily intended for the tracing of optical rays and will have a smaller storage capacity and will be less flexible in operation than the other machines here mentioned. The addition time will be 50 milliseconds and the multiplication time 1.5 seconds. The word length will be 31 digits. In Sweden a machine known

as BARK (Binar Automatish Rela-Kalkylator) is being constructed at the Tekniska Högskolan, Stockholm. This is a parallel binary machine with a word length of 32 digits and a floating binary point. It uses about 5500 relays and is programmed by means of a plug board on which a program of up to 840 orders may be plugged. Provision is made for the use of conditional orders and subroutines.

OTHER AIDS TO COMPUTATION

Addition and Subtraction on a Logarithmic Slide Rule

It does not seem to be generally known that the principle of addition logarithms can be applied to the use of an ordinary slide rule for adding. The process is, of course, not worthwhile if additions occur in isolation, but much time can be saved if additions occur in combination with multiplications or divisions, and if slide rule accuracy is sufficient.

Using the C and D scales, the sum $(a + b)$ can be found thus:

Set the index of C to the value of a on D (preferably choose $a > b$). Set the cursor to the value of b on D. Read the value of b/a on C, under the cursor. Form mentally $(1 + b/a)$ and set the cursor to this value on C. Read $(a + b)$ on D, under the cursor.

When additions are combined with another process, one of the terms can usually be arranged to appear on the D scale ready for addition, or the sum appearing on the D scale can be used there for the next process. For example $(ab + c)$ can be formed thus:

Set the cursor to a on D and move the slide so that b on CR lies below the cursor line; the index of C is now opposite ab on D. Move the cursor to the value of c on D, read c/ab on C under the cursor, add 1 mentally and set the cursor to $(1 + c/ab)$ on C. Read $(ab + c)$ on D, under the cursor.

Analogous methods apply to other combinations of operations involving addition, and subtractions can also be handled in a similar manner.

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14. *Polnoe Sobranie Sochinenii P. L. Chebysheva* [Complete Collection of Works by P. L. Chebyshev]. Volume 4: *Teoria Mechanismov* [Theory of Mechanisms]. Moscow and Leningrad, Academy of Sciences, 1948, 254 p. + portrait frontispiece, 16.5 × 25.5 cm.

This volume which is the fourth in a series of collected works of Chebyshev, contains fourteen articles on theory of mechanisms prepared by the author during the period of 1861–1888, a brief discussion of these articles, and a brief description of model mechanisms built by the author. The author's articles included in this volume were previously published in v. 1 and 2 of the first edition, 1899–1907, of his collected works [*MTAC*, v. 1, p. 440–441]. Five of the articles were published previously in various French publications. The author's article "Theory of Mechanisms Called Parallelograms," because of its mathematical nature was included in v. 2 which is devoted to the work in mathematical analysis.

Most of the articles are devoted to the mathematical derivation of parameters for design of four-bar linkages and harmonic transformer mechanisms which will best approximate the desired motion. Also included is a description of a computing machine (p. 158–160) with a continuous movement of the components rather than movement with discrete intervals such as used by common mechanical digital computers. The continuous motion is obtained by means of planetary gears.

The volume concludes with an article by I. I. ARTOBOLEVSKIĬ & N. I. LEVITSKIĬ on models of Chebyshev's mechanisms, which are preserved in the Leningrad Academy of Sciences. There are 25 of these mechanisms which are described and illustrated. The illustration of Chebyshev's "arithmometer" is disappointingly inadequate.

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15. J. G. L. MICHEL, "A nomogram for calculating extended terms," Institute of Actuaries Students' Soc., *Jn.* v., 8, 1948, p. 147–159.

Two nomograms are given for the calculation of endowment insurance.

16. F. J. MURRAY, "Linear Equation Solvers," *Quart. Applied Math.*, v. 7, 1949, p. 263–274.

17. G. H. ORCUTT, "A New Regression Analyzer," R. Stat. Soc., *Jn.*, Sec. A, v. 111, 1948, p. 54–70.

The analyzer described in this paper is based on units, each of which consists of a card reader and commutator. By means of this combination a time sequence of voltages $X_1, X_2, X_3, \cdots, X_n$ corresponding to the two-digit quantities punched on the cards is obtained. When a number of units are combined with suitable output circuits it is possible to obtain a variety of second degree expressions, for instance $\sum (X_i - \bar{X})\cdot(Y_i - \bar{Y})$ or $\sum X_i \cdot Y_{i+k}$, where k is a shift subject to the operator's control. The author discusses in detail the application of these expressions to statistical problems including those in which varying time lags between sequences are to be considered. The advantages of the use of the commutator over parallel operation consists in the simplicity of the associated circuits and the fact that the various sequences of voltages can be shown immediately on an oscillograph.

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NOTES

110. NEW FACTORIZATIONS OF $2^n \pm 1$.—In *MTAC*, v. 3, p. 496–7 we gave a proof of the primality of $(2^{92} + 1)/17$. Using the same methods we have now established the primality of

$$N_1 = (2^{79} + 1)/3 = 2014\ 87636\ 60243\ 81957\ 84363$$

and

$$N_2 = (2^{85} + 1)/(3\cdot 11\cdot 43691) = 26831\ 42303\ 60653\ 52611$$
$$N_3 = (2^{83} - 1)/167 = 579\ 12614\ 11327\ 56490\ 87721.$$

In the case of N_1, which is the 6-th largest known prime, we find that for $y = 2^{77}$,

$$3^y \equiv 534\ 45942\ 48656\ 40551\ 54581 \equiv W \pmod{N_1}$$

and that

$$W^2 \equiv -3 \pmod{N_1}. \quad \text{Hence } 27^{N_1} \equiv 27 \pmod{N_1}.$$

The largest prime factor of $N_1 - 1$ is $p = 22366891 = (N_1 - 1)/m$. Furthermore

$$3^m - 1 \equiv 591\ 99625\ 59867\ 68206\ 78419 \pmod{N_1},$$

which is prime to N_1. From this it follows, by LEHMER's theorem, that all the prime factors of N_1 are of the form $px + 1$. Combining this with the fact that they are also of the form $158x + 1$ we obtain

$$dx + 1 = 3533968778x + 1.$$

Now if N_1 were composite we could write $N_1 = (dm + 1)(dn + 1)$ with $mn \neq 0$. Using the reasoning of the footnote on p. 497 it follows that the remainder on division of N_1 by d^2 would be less than 10^{14}, whereas it is greater than $3.6 \cdot 10^{18}$. Hence N_1 is not composite.

In the case of N_2, the proof was more difficult. In the first place it was found that if $q = 3 \cdot 11 \cdot 43691$ and if $Q = 3^q$, then

$$Q^{N_2} \equiv Q \pmod{N_2},$$

so that N_2 behaves like a prime. Next it was found that

$$N_2 - 1 = 2 \cdot 3 \cdot 5 \cdot 17 \cdot 257M = F \cdot M,$$

where

$$M = 20471\ 06358\ 13423.$$

The number M was in turn tested for primality using the facts that $3^M \equiv 3 \pmod{M}$ and that $M - 1$ is divisible by the prime 10235291.

The primality of N_2 now follows as before from the primality of M and the fact that

$$3^F - 1 \equiv 24823\ 70333\ 63136\ 14240 \pmod{N_2},$$

is prime to N_2.

With the exceptions of $2^{71} + 1$ and $2^{89} + 1$, the first 100 numbers of the form $2^n + 1$ are now completely factored.

The primality of N_3 follows in a similar way. First of all N_3 behaves like a prime, since it was found that if $a = 2^{83}$, then

$$3^a \equiv -3^{84} \pmod{N_3}.$$

Next $N_3 - 1$ is divisible by 383 and 4049. By using Lehmer's theorem it was proved that the possible prime factors of N_3 are all of the forms $383y + 1$, $4049z + 1$, $166w + 1$, and hence of the form

$$257427322x + 1.$$

The same argument as applied to N_1 finally establishes the primality of N_3. This makes the 15-th composite MERSENNE number to be completely factored.

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111.—Electronic Computers and the Analysis of Stochastic Processes.—Hartree [1] has remarked on the impact of modern calculating machines upon mathematical analysis: by suitably recasting the mathematical treatment of a problem we may profit from the capacity and speed of these machines to get quick solutions of long or otherwise intractable enquiries. Stochastic processes supply a noteworthy illustration of the mathematician's need to think in terms of numerical methods.

Leslie [2] considers the deterministic growth of animal populations; and by formal mathematics he reduces the problem of specifying the growth-rate, knowing the birth and death mechanism, to the determination of the dominant latent root of a matrix. This solves the problem analytically; but to solve it numerically we must calculate this latent root. Duncan & Collar [3] give the appropriate computing technique. To study their method is instructive: for it is identical with Leslie's analysis—that is, they take a given population (represented by a vector), subject it to a given birth and death process (represented by a matrix), and deduce the value of the dominant latent root from the observed growth-rate of the vector. This exemplifies the following common situation. There is a practical problem A, stated in terms of practical data B. To solve A, we set up C, the mathematical model of A; and by formal analysis deduce D, the solution of C. To compute D, however, for the given conditions B, we set up E, the numerical model of A conforming to B; and solve E by numerical methods. Evidently the construction of C and its reduction to D are unnecessary steps if we merely wish a solution D. This is not to stigmatize C as useless; for a formal symbolic solution often affords insight into the structure of a problem, and sometimes is more tractable than an entirely numerical resolution of E.

Situations of this kind will probably occur in abundance in the analysis of stochastic processes. To take one of the simplest instances, corresponding to the deterministic description of population growth, there is a more realistic stochastic description with a formal solution in terms of integral equations; and the position here is more extreme than in the example previously cited, because the numerical solution of these equations is more elaborate than the evaluation of a dominant latent root.

Stochastic processes are a relative innovation still certainly in their infancy: but it is clear from three fundamental papers [Bartlett,[4] Kendall,[5] and Moyal [6]] that these processes, ranging as they do over a huge field of applications from epidemiology to atomic physics, will prove singularly important in the near future. It is proper to ask how far mathematical methods, and particularly numerical methods, are girded up to meet the coming demands from this quarter.

From a very general aspect, the problems comprise a set of arithmetical operations applied to random space-time functions, yielding answers whose complete specification involves statistical distributions. A possible direct attack—direct in the same sense as the Duncan-Collar method is direct when applied to Leslie's problem—is to feed in as data a sample from the random function aggregate, subject each member of the sample to the relevant arithmetical operations, and enumerate the results. This has the merit of simplicity; and the apparent drawback, that a large sample is probably needed to give an adequate representation of the output distributions, can be countered by the electronic computer's distinctive facility of

performing routine arithmetic operations upon large masses of data, provided that the rate of supply of data is commensurate with the operating speed of the computer. Is it then feasible to generate random function samples electronically within the computer? At any rate, at first sight, this idea seems promising. The noise and shot effects of a thermionic valve furnish random functions. A random pulse/blank train in a computer's binary decimal sequence generates a rectangular distribution, and can (at least theoretically) lead to an integrated Fourier series whose coefficients are distributed normally and independently over the complex domain [PALEY & WIENER [7]].

This is not however the place to enter into details, even were they less speculative: but it is a matter for consideration whether stochastic processes could be analyzed in the direct fashion suggested on an electronic computer, and, if so, whether they will be pervasive enough to warrant building a special unit into the computer to generate random functions; and this note will have served its purpose if it provokes research on this issue at the present opportune juncture, when a number of electronic computers are projected or under construction or in their developmental stages in various parts of the world.

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[7] R. E. A. C. PALEY & N. WIENER, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc., *Colloq. Pub.* no. 19, New York, 1934.

QUERY

33. LENHART TABLES.—As a supplement to the final number, 6, Nov. 1838, of *The Mathematical Miscellany*, v. 1, edited by CHARLES GILL (1805–1855), is a 16-page pamphlet, with its own title-page, as follows: *Useful Tables relating to Cube Numbers, Calculated and arranged* by WILLIAM LENHART, *York, Penn. Designed to accompany his general investigation of the equation* $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, *published in the Mathematical Miscellany, vol. 1, page 114; and by him through his friend, Professor C. Gill, presented to the Library of St. Paul's College, Flushing, Long Island, May 4th, 1837*. As indicated in D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, 1941, p. 64, this "rare table" "gives, for more than 2500 integers $A < 100\,000$, solutions of $x^3 + y^3 = Az^3$ in positive integers." On the back of the title page of this pamphlet is the following: "Besides the tables given

here, the manuscript copy compiled with so much labor and care, by Mr. Lenhart, includes a Table,

'Containing a variety of Numbers between 1 and 100,000, and the roots, not exceeding two places of figures, of two cubes, to whose difference the numbers are respectively equal'; together with a Table,

'Exhibiting the roots of three cubes to satisfy the indeterminate equation

$$x^3 + y^3 + z^3 = A,$$

for all values of A, from 1 to 50 inclusive.'

"Both these tables are extremely curious, and are open to inspection of all who may wish to consult them. They are lodged in the library of St. Paul's College."

This was probably written by Gill.

Numbers I–IV of the *Mathematical Miscellany* were published at the Flushing Institute, which had become St. Paul's College when numbers V–VIII (1838–1839) were published. But by 1844 this College had ceased to function, and hence also its Library, no doubt.

Can any one tell us if the above mentioned ms. tables of Lenhart[1] (1787–1840) have been preserved in any library or have ever been published?

R. C. A.

[1] Lenhart made a number of excellent contributions to the *Mathematical Miscellany* and his name is mentioned several times in L. E. DICKSON, *History of the Theory of Numbers*, v. II; *Diophantine Analysis*, Washington, 1920. Many personal details are given in [S. TYLER], "The life of Lenhart the mathematician," *The Biblical Repertory and Princeton Review*, v. 13, 1841, p. 394–416. The name of the author of this anonymous article was taken from the *Index Volume*, 1871, of the *Repertory*. See also W. S. NICHOLS, "William Lenhart, the American Diophantist, potential actuary and mathematical testator of Professor Charles Gill," Actuarial Soc. Amer., *Trans.*, v. 21, 1920, p. 118-122, 124; note by W. A. HUTCHESON, p. 122–124. Also CALVIN MASON, York [Pa.] *Gazette*, 14 Sept. 1841.

That the Yorkshireman Gill, mathematician, and the first actuary in America (he prepared an Actuary's Report on the experience of The Mutual Life Insurance Co. of New York), does not appear in the *Dictionary of American Biography* is surely an oversight. See E. MCCLINTOCK, Actuarial Soc. Amer., *Trans.*, v. 14, 1913, p. 9-16, 212–237; v. 15, 1914, p. 11–39 + portrait, 228–270. "Historical sketch of the life of CHARLES GILL, Esq., late actuary of the Mutual Life Insurance Company of New York," Institute of Actuaries, *Assurance Mag.*, v. 6, 1857, p. 216–227. C. WALFORD, *The Insurance Cyclopaedia*, v. 5, London, 1878, p. 394. *The International Insurance Encyclopedia*, New York, v. 1, 1910, p. 313. D. E. SMITH & J. GINSBURG, *A History of Mathematics in America before 1900*, Chicago, 1934, p. 89, 98–99. S. NEUMARK, "Note on the life of Charles Gill," *Scripta Mathematica*, v. 2, 1934, p. 139–142.

QUERIES—REPLIES

43. INTEGRAL EVALUATIONS (Q 22, v. 2, p. 320).—In partial reply we may note that the integral

$$I(t) = \int_0^t \cos (a_0 + a_1x + a_2x^2)dx,$$

where the a's are real, may be evaluated in terms of the so-called FRESNEL integrals

$$C(u) = \int_0^u \cos (\tfrac{1}{2}\pi\theta^2)d\theta, \qquad S(u) = \int_0^u \sin (\tfrac{1}{2}\pi\theta^2)d\theta,$$

tables of which are listed in *MTAC*, v. 1, p. 250, v. 2, p. 336, v. 3, p. 417, 467, 479, v. 4, p. 24, 30.

We may suppose that a_2 is positive so that $a_2 = a^2$. Completing the square and using the cosine addition theorem gives

$$\begin{aligned} a(2/\pi)^{\frac{1}{2}}I(t) &= [C(bt + c) - C(c)]\cos\delta \\ &\quad - [S(bt + c) - S(c)]\sin\delta, \end{aligned}$$

where

$$\delta = a_0 - a_1^2/(4a^2), \qquad b = a(2/\pi)^{\frac{1}{2}}, \qquad c = (2\pi)^{-\frac{1}{2}}a_1/a.$$

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CORRIGENDA

V. 1, p. 184, l. 20 *for* 9D *read* exact.
V. 1, p. 336, 468 *for* Eschbach *read* Eshbach.
V. 3, p. 457, l. 19 *for* a_{1i} *read* a_{1j}.
V. 3, p. 458, l. 2, *for* $a_{ij} - a_{i1}a_{1i}/a_{ii}$ *read* $a_{i1}a_{1j}/a_{11}$.

[illegible]

[illegible]

[illegible]

[illegible]